9 Sequences and Series

The **TI-89/92** has graphical, algebraic, and numerical capabilities that lend themselves to the study of sequences and series. This chapter is an introduction to these capabilities and a supplement to Chapter 12 of Stewart's *Calculus*.

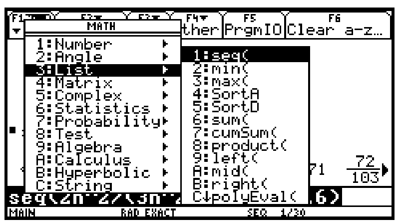
9.1 Sequences

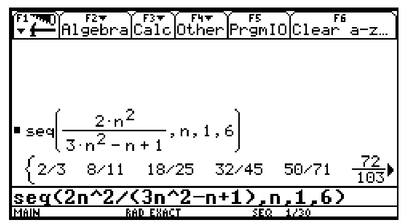
This section is devoted to the ways in which the **TI-89/92** enables us to study sequences. First we'll look at the **TI-89/92**'s ability to generate sequences and find limits symbolically.

• EXAMPLE 1. Consider the sequence defined by

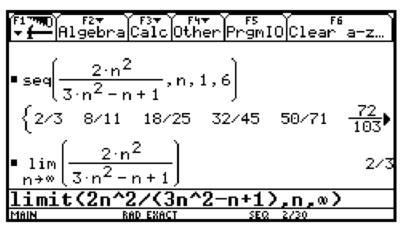
$$a_n = \frac{2n^2}{3n^2 - n + 1}, \quad n = 1, 2, 3, \ldots \ .$$

The **seq()** command, which is found under **List** in the **MATH** menu (**[2nd][5]**), can be used to display a number of terms of the sequence. Exact values are displayed if **EXACT** is selected as the **Exact/Approx MODE**.

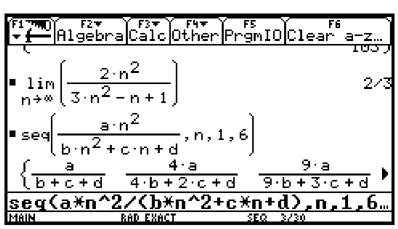


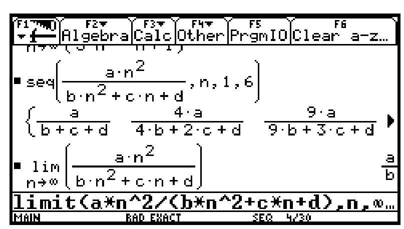


The **limit()** command, found in the **Calc** menu (**F3** from the Home screen), computes the limit of the sequence.



This also works with sequences whose terms involve symbolic parameters, as seen in the following screens.



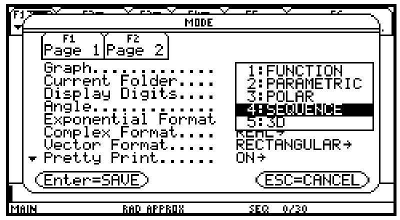


Plotting sequences. Sequences can be plotted easily on the **TI-89/92**.

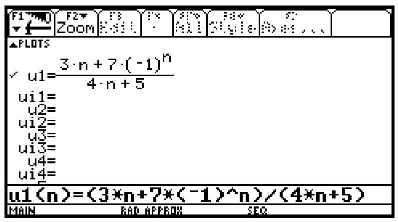
• EXAMPLE 2. Suppose that we wish to study the sequence defined by

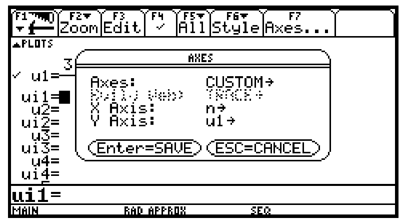
$$a_n = \frac{3n + 7(-1)^n}{4n + 5}, \quad n = 1, 2, 3, \ldots \ .$$

The first step in plotting the sequence is to press the **MODE** key and select **SEQUENCE** as the **Graph** mode.

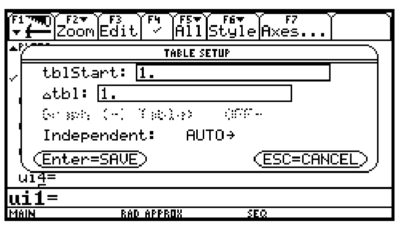


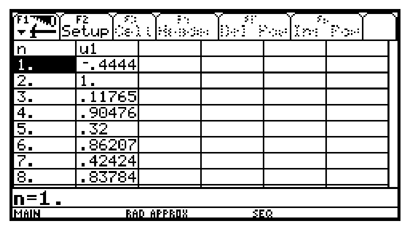
Then press ◇**Y=** to bring up the **Y=** Editor. There, enter the formula for a_n as **u1**. Also from the **Y=** Editor, press **F7** and specify **n** as the variable for the **X Axis** and **u1** as the variable for the **Y Axis**.



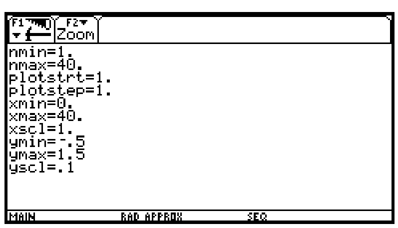


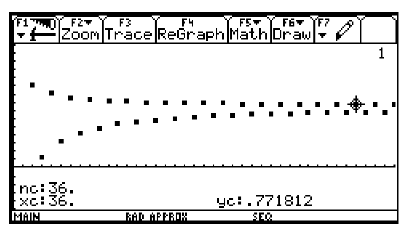
In order to get an idea what window bounds are appropriate, we can examine the terms in the sequence as a list. So press ◇**TblSet** and set each of the parameters **tblStart** and **∆tbl** equal to 1, and then press ◇**TABLE**.



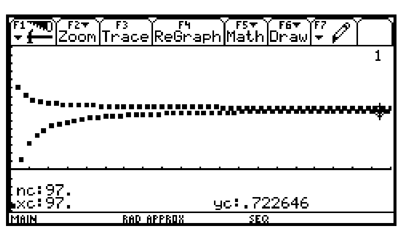


From the values in the table, it looks as if −.5 to 1.25 would be good choices for **ymin** and **ymax**. So press ◇**WINDOW** and enter window parameter values as shown below. Here we are plotting the first forty terms in the sequence. Then press ◇**GRAPH**. Notice that we have used **Trace** (**F3**) to display the value of the 36th term.

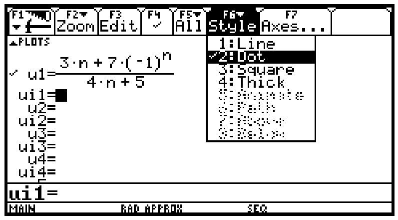


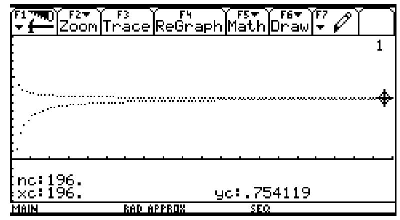


To better see the long-term behavior of the terms in the sequence, we can replot the sequence after setting **nmax** and **xmax** to 100.



Finally, let's look at a plot of the first 200 terms. But first we'll change the plot **Style** to **Dot**.





All of this provides numerical and graphical evidence that the limit of the sequence is 3/4. To prove that this is indeed the case, first observe that

$$\frac{3n-7}{4n+5} \leq \frac{3n+7(-1)^n}{4n+5} \leq \frac{3n+7}{4n+5}$$

for all $n \geq 1$. Now the squeeze theorem tells us that

$$\lim_{n\to\infty} \frac{3n+7(-1)^n}{4n+5} = \frac{3}{4},$$

because

$$\lim_{n\to\infty} \frac{3n-7}{4n+5} = \lim_{n\to\infty} \frac{3n+7}{4n+5} = \frac{3}{4}$$

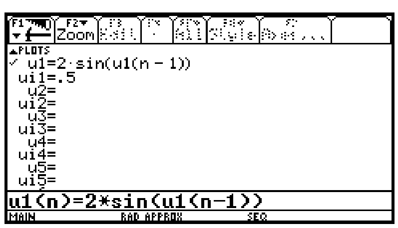
by application of l'Hôpital's Rule to the functions $f(x) = (3x-7)/(4x+5)$ and $g(x) = (3x+7)/(4x+5)$.

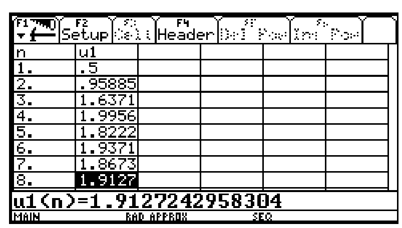
Recursive sequences. Often the n^{th} term of a sequence depends not upon n, but upon previous terms in the sequence. Such a sequence is called a *recursive* sequence.

• EXAMPLE 3. The sequence $\{a_n\}_{n=1}^{\infty}$ defined by

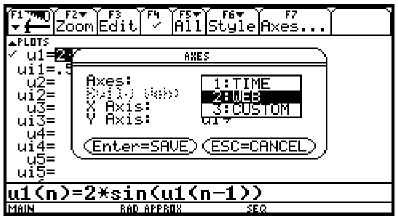
$$a_1 = .5$$
$$a_n = 2\sin a_{n-1},\ n = 2, 3, 4, \ldots$$

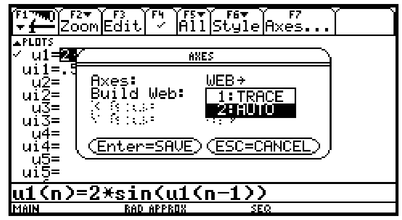
is a recursive sequence. In the **Y=** Editor, we enter a_n as **u1(n)** and then press ◇**TABLE** to view the first few terms.



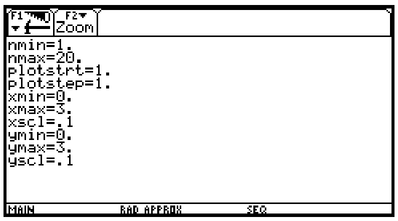


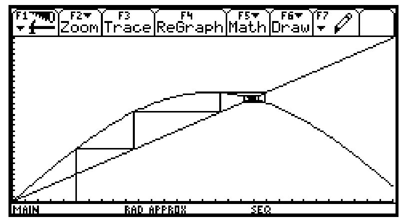
In fact, scrolling down the table a bit suggests that the sequence is converging to a limit that is approximately 1.8955. The generation of sequences such as this, in which $a_n = g(a_{n-1})$, with some given continuous function g and some given value of a_1, is often called *functional iteration*, or *fixed-point iteration*. There is a special graphical device, called a *web plot*, for visualizing such a sequence. To set up a web plot on the **TI-89/92**, press **F7** from the **Y=** Editor, and choose **Axes** = **WEB** and, for now, **Build Web** = **AUTO**.



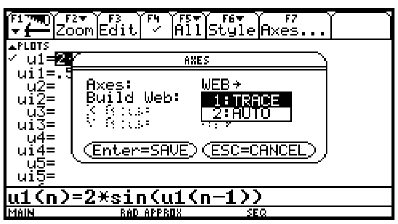


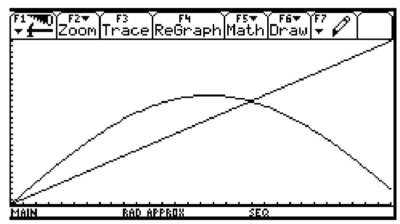
Now we set appropriate window variables and plot the graph.



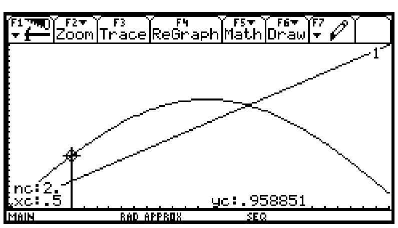


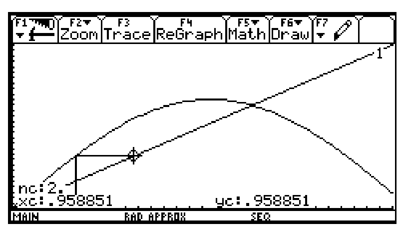
Note that the curve in the picture is the graph of $y = 2\sin x$ and the line is $y = x$. To get a better feeling for what this web plot is about, let's change to **Build Web** = **TRACE** in the **Axes** menu (**F7**) of the **Y=** Editor. Then pressing ◇**GRAPH** only shows the graphs of $y = 2\sin x$ and $y = x$.



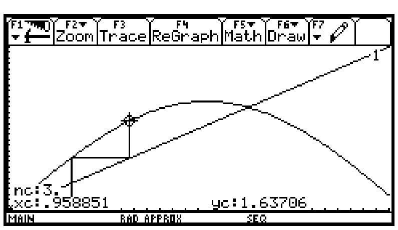


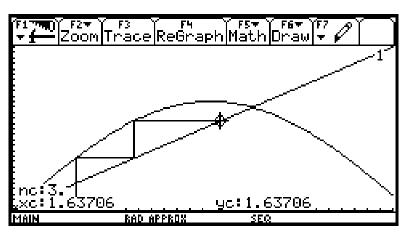
Now select **Trace** (**F3**) and press **[⇒]** once and then again.



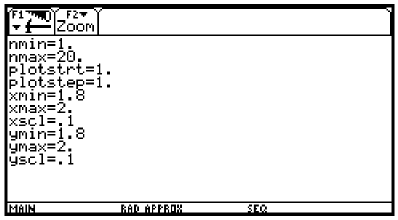


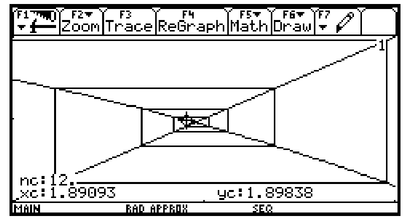
The first of these shows the computation of $a_2 = .958851$ from $a_1 = .5$, and the second shows the location of a_2 on the x-axis. Now press **[⇒]** twice more to find a_3 on both axes.





Continuing in this way, you will begin to see the "web" approach the point of intersection of the two graphs. It is also interesting to zoom in a bit to observe the behavior of the sequence once terms begin to get fairly close to the limit.





Our investigations so far indicate that

$$\lim_{n\to\infty} a_n \approx 1.8955.$$

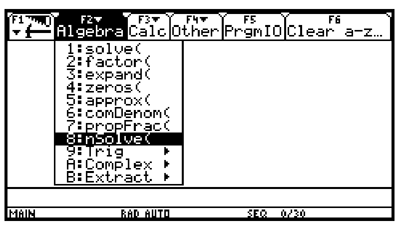
Let x^* denote the exact value of this limit. Since

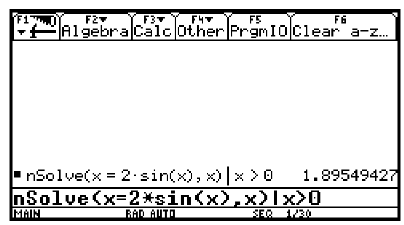
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n-1} = x^*$$

and $a_n = 2\sin a_{n-1}$ for all n, it must be true that

$$x^* = 2\sin x^*;$$

that is, the limit x^* is a solution of the equation $x = g(x)$, where $g(x) = 2\sin x$—that is, a *fixed point* of the function g. (Note that the points of our web plot converged to the point of intersection of the two graphs.) Let's check this by solving $x = 2\sin x$ with **nSolve()**.





This is an example of a very important use of sequences. Exact solutions of many equations, such as $x = 2\sin x$, simply cannot be found. Thus we resort to some approximation procedure which amounts to generating a sequence that converges (we hope) to a solution. Provided that the sequence does converge, by computing enough terms in the sequence we can approximate the exact solution to any desired accuracy.

Newton's Method revisited. Recall that Newton's Method for approximating a solution of $f(x) = 0$ is described by

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}, \quad n = 1, 2, 3, \ldots\ ,$$

where x_0 is some chosen initial approximation to the solution. This procedure generates a recursive sequence, since $x_n = g(x_{n-1})$ for $n \geq 1$, where

$$g(x) = x - f(x)/f'(x).$$

Note that if $x_n \to x^*$ as $n \to \infty$ and if $f'(x^*) \neq 0$, then

$$x^* = x^* - \frac{f(x^*)}{f'(x^*)},$$

which implies that $f(x^*) = 0$.

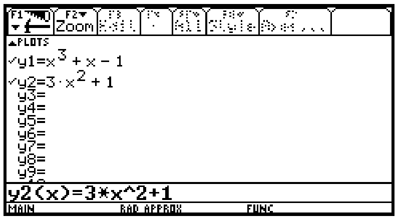
• Example 4. Consider the problem of approximating the solution of

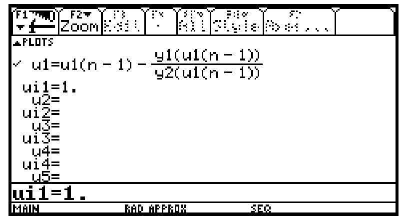
$$x^3 + x - 1 = 0.$$

Examination of the graph of the function $f(x) = x^3 + x + 1$ shows that there is exactly one solution and that this solution is between .5 and 1. So we'll take $x_0 = 1$ and use Newton's method, which here takes the form

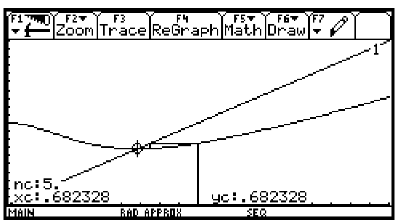
$$x_n = x_{n-1} - \frac{x_{n-1}^3 + x_{n-1} - 1}{3x_{n-1}^2 + 1}, \quad n = 1, 2, 3, \dots \ .$$

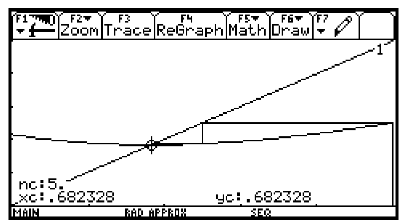
With **Graph MODE = FUNCTION**, enter $f(x)$ as y1 and $f'(x)$ as y2. Then switch to **Graph MODE = SEQUENCE**, and enter the Newton iteration formula as **u1**.





Press ◇**TABLE** to see the very rapid convergence of the sequence to a limit $x^* \approx .68233$. Indeed, notice that we have at least five decimal places of accuracy after only four iterations. ◇**GRAPH** shows the following picture on the left with window bounds of 0 to 2 on each axis and the picture on the right with window bounds of .5 to 1 on each axis.





The curve in each of the pictures above is the graph of

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Notice that the extremely rapid convergence of the sequence is due to the fact that the graph of g has a horizontal tangent at the point of intersection. One of the exercises that follow will ask you to verify that this is a general property of Newton's Method.

Exercises

For Exercises 1–5, plot each sequence and try to determine whether the sequence has a limit as $n \to \infty$ and, if so, estimate the value of the limit or make a conjecture about its exact value. Then have your **TI-89/92** compute the limit with its **limit()** command.

1. $a_n = \dfrac{2 + (-1)^n n^2}{n^2 - 3n + 4}$
2. $a_n = \dfrac{\ln(n)}{n^{1/3}}$
3. $a_n = \dfrac{n!}{200^n}$
4. $a_n = \dfrac{2n^3 - 6n^2 + 15}{n^4 + 18n^3 - 6}$
5. $a_n = (\cos^2 3n)^{1/n}$

In Exercises 6–10, make a web plot of each recursive sequence and try to determine whether the sequence has a limit as $n \to \infty$ and, if so, estimate the value of the limit or make a conjecture about its exact value. When possible, compute the exact value of the limit. In all cases, describe as well as you can the long-term behavior of the sequence.

6. $a_n = 1 + a_{n-1}/3, \quad a_1 = 0$

7. $a_n = 1 + a_{n-1}/3, \quad a_1 = 3$

8. $a_n = 2a_{n-1} - 3, \quad a_1 = 2.99$

9. $a_n = e^{-a_{n-1}}, \quad a_1 = 1$

10. $a_n = a_{n-1}(3 - a_{n-1}), \quad a_1 = 1.5$

In Exercises 11–15, use Newton's Method to approximate the smallest positive solution of the given equation to at least six decimal places.

11. $\sin^2 x - \cos(x^2) = 0$

12. $x - \cos x = 0$

13. $x = e^{-x}$

14. $\tan x = x$

15. $x^5 - x^4 = 1$

16. Let $g(x) = x - f(x)/f'(x)$, where f is a given twice-differentiable function.
   a) Show that $g'(x) = f(x)f''(x)/f'(x)^2$.
   b) Conclude that if $f(x^*) = 0$ and $f'(x^*) \neq 0$ (i.e., x^* is a simple root of f), then $g'(x^*) = 0$.
   c) Describe the effect of $g'(x^*) = 0$ on the web plot of the Newton's Method iteration.

9.2 Series

For any sequence $\{a_n\}_{n=1}^{\infty}$, there is an associated *sequence of partial sums* $\{s_n\}_{n=1}^{\infty}$ defined by

$$s_n = \sum_{k=1}^{n} a_k.$$

• Example 1. If $a_n = 1/n^3$, then

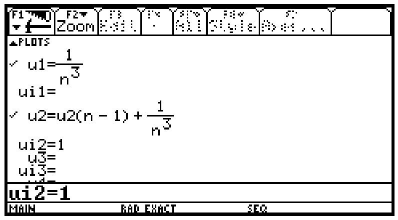
$$\begin{aligned} s_1 &= 1, \\ s_2 &= 1 + \frac{1}{8} = \frac{9}{8}, \\ s_3 &= 1 + \frac{1}{8} + \frac{1}{27} = \frac{251}{216}, \\ s_4 &= 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} = \frac{2035}{1728}, \quad \dots \end{aligned}$$

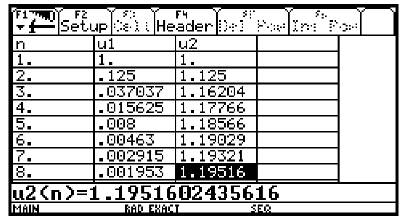
Notice that the terms of any such sequence of partial sums will satisfy

$$\begin{aligned} s_1 &= a_1, \\ s_n &= s_{n-1} + a_n, \; n = 2, 3, 4, \dots \; . \end{aligned}$$

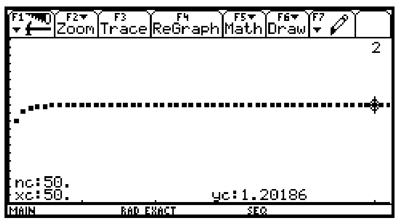
This allows very efficient calculation of the partial sums.

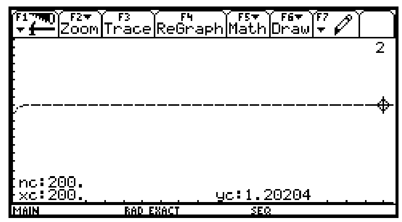
• Example 2. Consider again the sequence $a_n = 1/n^3$. To investigate the behavior of the partial sums, let's enter a_n as **u1** and s_n as **u2** in the **Y=** Editor and construct a table of values.



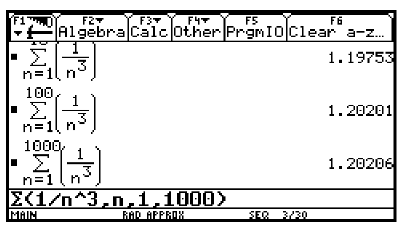


Values further down the table suggest that the partial sums are converging to a limit approximately equal to 1.2. To estimate this (supposed) limit more accurately, it is more efficient to plot the sequence of partial sums and use either **Trace** (**F3**) or the **Value** function from the **Math** menu (**F5-1**).





An alternative approach is to use the $\sum$**()** operator from the **Calc** menu on the Home screen. (Be sure to switch **Exact/Approx MODE** to **APPROXIMATE**.)



The numbers we are seeing suggest that this sequence of partial sums is a convergent sequence with a limit approximately equal to 1.202. However, this is an area where one must be very careful about such claims.

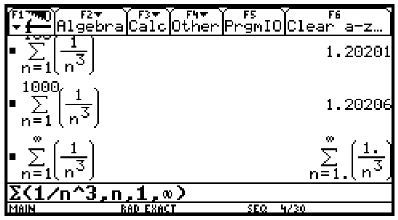
Given a sequence $\{a_n\}_{n=1}^{\infty}$, the limit of the associated partial sums is called an *(infinite) series* and is denoted by

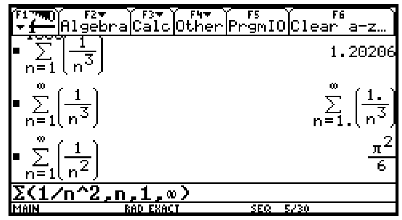
$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k.$$

For example, our preceding investigation suggests that

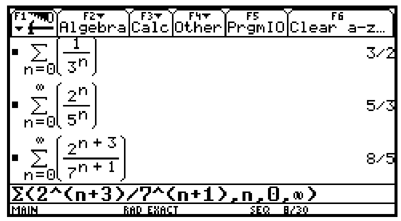
$$\sum_{k=1}^{\infty} \frac{1}{k^3} \approx 1.202.$$

The **TI-89/92** can compute the exact value of certain series. This is done, again, with the $\sum$**()** operator. As we see below, $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is not a series that the **TI-89/92** can compute. However, $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is. (Switch **Exact/Approx MODE** back to **EXACT**.)





Among the series that the **TI-89/92** can compute exactly are *geometric series*.



Convergence. It is often difficult to determine whether a sequence converges by numerical and graphical means. For example, it would be very difficult by graphical or numerical investigations to determine whether either the *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

or the p-series (with $p = 1.01$)

$$\sum_{n=1}^{\infty} \frac{1}{n^{1.01}},$$

is convergent. (The harmonic series is known to diverge, while p-series with $p > 1$ are known to converge.) For this reason, we need analytical tests to determine whether a series converges. Moreover, when dealing with a convergent series, it is very important to have an estimate of the error when estimating the series with a partial sum.

There are five primary tests for convergence. These are the integral test, the comparison test, the limit-comparison test, the ratio test, and the root test. Because it also provides a useful error estimate, we will illustrate only the ...

Integral test: *Let $a_n = f(n)$, where f is a positive, continuous, and decreasing function on the interval $[1, \infty)$. Then*

$$\sum_{k=1}^{\infty} a_k \quad \text{converges if and only if} \quad \int_1^{\infty} f(x)dx \quad \text{converges.}$$

Moreover, we have the error estimate

$$\int_{n+1}^{\infty} f(x)dx \le \sum_{k=n+1}^{\infty} a_k \le \int_n^{\infty} f(x)dx$$

for the partial sum s_n.

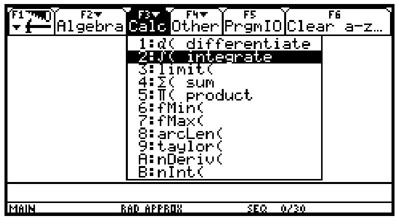
• Example 3. Consider the series

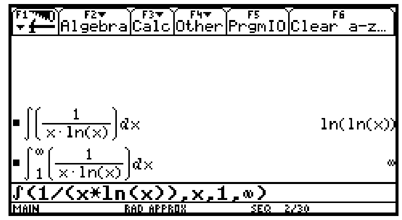
$$\sum_{k=1}^{\infty} \frac{1}{n \ln n}.$$

According to the integral test, this series converges if and only if the improper integral

$$\int_1^\infty \frac{dx}{x \ln x}$$

converges. The $\int$**()** operator from the **Calc** menu lets us evaluate this integral easily. We see that an antiderivative of $(x \ln x)^{-1}$ is $\ln(\ln x)$, which clearly approaches ∞ as $x \to \infty$. Further verification is provided by the computation of the improper integral.





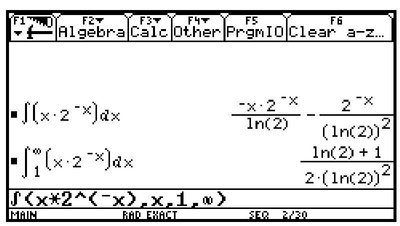
So we see that the series in question diverges.

• Example 4. Consider the series

$$\sum_{k=1}^{\infty} \frac{n}{2^n} \ .$$

According to the integral test, the convergence or divergence of the series is determined by that of the improper integral

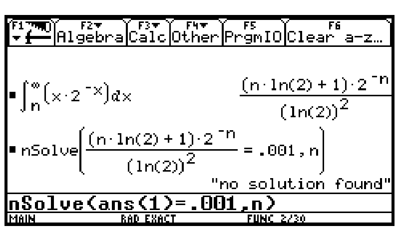
$$\int_1^\infty \frac{x}{2^x} dx.$$

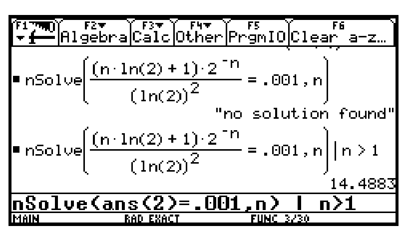


Thus we see that the series converges. Now suppose we wish to estimate the value of this series to within 0.001. How many terms must we sum in order to obtain that accuracy? According to the error estimate provided by the integral test, we need to find n such that

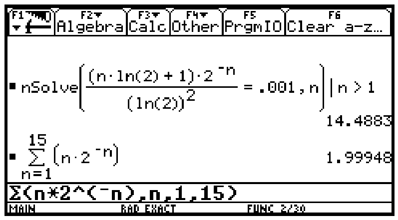
$$\int_n^\infty \frac{x}{2^x} dx \leq .001.$$

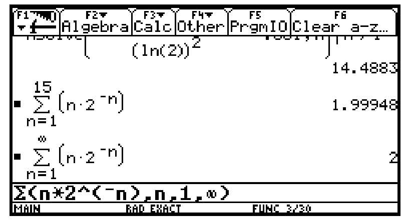
So first we compute $\int_n^\infty \frac{x}{2^x} dx$ in terms of n; then we set this equal to .001 and try to solve for n using **nSolve** from the **Algebra** menu. Note that **nSolve** is not successful until we restrict its search to $n > 1$, when it returns $n = 14.4883$.





So $n = 15$ terms will provide the accuracy we desire. Computation of that partial sum suggests that the exact value of the series may be 2. In fact, the **TI-89/92** can compute the exact value of this sum, which is indeed equal to 2.





Power series. A power series, about the number a, is a function f defined, for some given sequence of coefficients $\{c_n\}_{n=0}^{\infty}$, by

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

The domain $\mathcal{D}_f$ of such a function is the set of all x for which the series converges. There are three possibilities for the domain $\mathcal{D}_f$. Either:

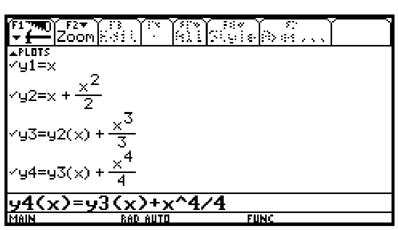
(i) $\mathcal{D}_f = \{a\}$,
(ii) $\mathcal{D}_f = (-\infty, \infty)$, or
(iii) $\mathcal{D}_f$ is a bounded interval centered at a.

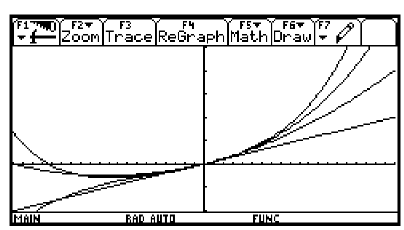
In cases (ii) and (iii), the domain is called the *interval of convergence.* In case (iii) it may be an open interval, a closed interval, or it may contain one of its two endpoints. Half the length of this interval is called the *radius of convergence* of the power series. In case (i), we say that the radius of convergence is zero; while in case (ii), we say that the radius of convergence is ∞.

• Example 5. Consider the power series

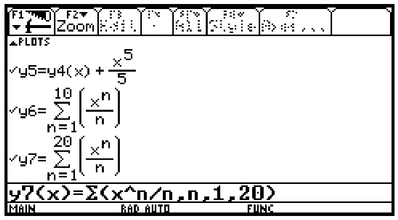
$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \, .$$

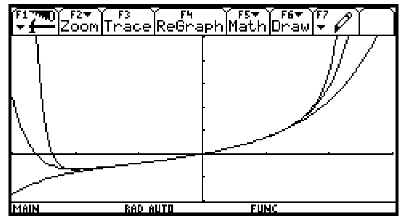
The ratio test shows that the series converges (absolutely) for $-1 < x < 1$. The series diverges when $x < -1$ or $x \geq 1$ and is a convergent alternating series when $x = -1$. By graphing several of the polynomials obtained by truncating the series, it is possible to visualize the convergence of the power series as a sequence of functions. Let's first graph the first through the fourth partial sums of the series; i.e., the first through the fourth degree approximations to the series, on the interval $[-2, 2]$.



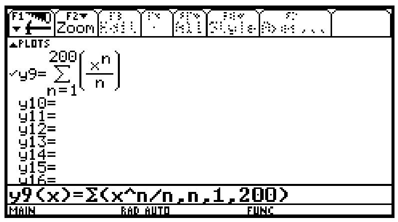


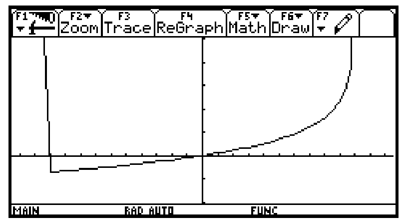
What should be noticed here is that the graphs are very close to each other near $x = 0$. Now let's graph the the fifth, tenth, and 20th degree approximations to the series on the interval $[-1.5, 1.5]$.





Finally, we'll graph the 200th partial sum on the interval [−1.25, 1.25]. (***Warning***: It takes the **TI-89/92** several minutes to do this.)





Notice that the derivative of function $f(x)$ in this example is

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} x^{n-1} \\ &= \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for } -1 < x < 1\ . \end{aligned}$$

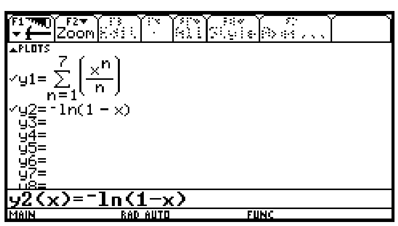
Also, $f(0) = 0$. Therefore we can conclude that f has the "*closed form*"

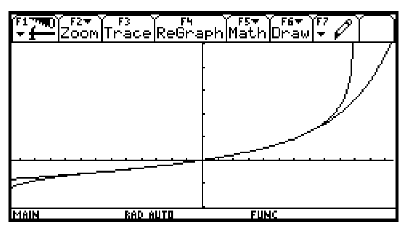
$$f(x) = -\ln(1-x) \quad \text{for } -1 < x < 1\ ;$$

that is,

$$-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{for } -1 < x < 1\ .$$

Let's look at the graph of this function, together with, say, the seventh partial sum of the power series for comparison.



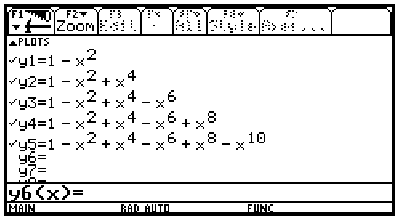


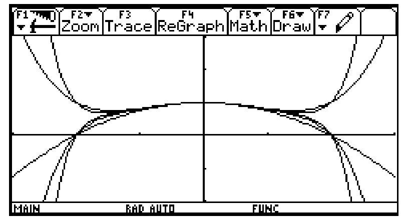
Note that the graph of $-\ln(1-x)$ is the one with the vertical asymptote at $x = 1$.

• Example 6. Consider the power series

$$f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

The ratio test reveals that the series converges (absolutely) for $|x| < 1$ and diverges for $|x| > 1$. It also clearly diverges if $x = \pm 1$. So let's first plot the second through the fifth partial sums on the interval [−1.5, 1.5].

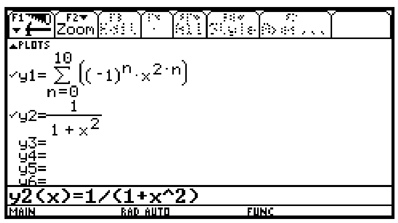


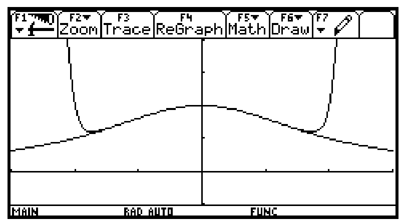


From what we know about the geometric series, it is not difficult to see the closed form

$$\sum_{n=0}^{\infty}(-1)^n x^{2n} = \frac{1}{1+x^2} \quad \text{for } -1 < x < 1.$$

So let's plot the eleventh partial sum together with $y = (1+x^2)^{-1}$, again on the interval $[-1.5, 1.5]$.





This graph does a very good job of illustrating the convergence of partial sums of a power series. In this case, the function represented by the power series is defined and bounded for all x, but the power series converges to it only on the power series's interval of convergence.

Exercises

For each series in 1–5, plot the sequence of partial sums and estimate the value of the series.

1. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$
2. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{2^n}$
3. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^n}$
4. $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}$
5. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{\pi}{3}\right)^{2n}$

For each series in 6–8, use the integral test error estimate to determine how many terms, when summed, will estimate the series to within 0.0001.

6. $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$
7. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$
8. $\displaystyle\sum_{n=1}^{\infty}\frac{\ln(1+n^2)}{n^2}$

For each power series in 9 and 10,

a) find the interval of convergence with the ratio test;
b) plot the first five and the tenth partial sums of the series on an appropriate interval;
c) find the closed form of the series and plot the graph on its interval of convergence.

9. $\displaystyle f(x)=\sum_{n=1}^{\infty}\frac{(-1)^{(n+1)}x^{2n}}{n}$
10. $\displaystyle f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$

9.3 Taylor polynomials

The n^{th} degree Taylor polynomial about a point $x = a$ for a function f that is n times differentiable in an open interval containing a is given by

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k,$$

where $f^{(k)}$ denotes the kth derivative of f. If f has derivatives of all orders in an open interval containing $x = a$, then its Taylor polynomials are partial sums of the power series

$$f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$$

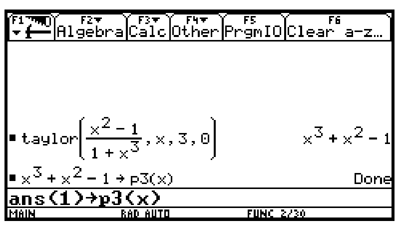
where the coefficients c_k are given by $c_k = f^{(k)}(a)/k!$. We call this series the *Taylor series* for f about $x = a$.

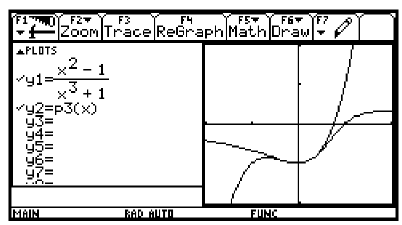
The **TI-89/92** has a built-in function for computing Taylor polynomials. It is the function **taylor()**, in the **Calc** menu.

• Example 1. Consider the function

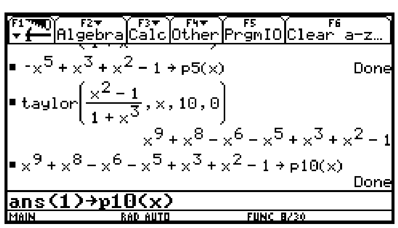
$$f(x) = \frac{x^2-1}{x^3+1}.$$

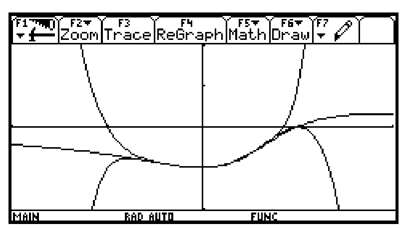
We'll find the third degree Taylor polynomial $T_3(x)$ about $x = 0$. In order to graph $T_3(x)$ along with $f(x)$, we assign the result of **taylor()** to the variable p3(x).





Finally, let's graph $f(x)$ along with $T_5(x)$ and $T_{10}(x)$. (Note that $T_{10}(x)$ is actually a polynomial of degree nine.)





The remainder term. Taylor polynomials provide an effective way of approximating a function locally about a given point $x = a$. According to *Taylor's Theorem*, the remainder $R_n(x)$ in the approximation of $f(x)$ by $T_n(x)$ is given by the formula

$$f(x) - T_n(x) = R_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x-a)^{n+1},$$

where ξ_x is some number between x and a, provided that f is $(n+1)$-times differentiable on some interval containing a and x.

• Example 2. Suppose that we approximate $\sin(.3)$ by computing the value of the cubic Taylor polynomial about $x = 0$ at .3:

$$\sin(.3) \approx .3 - \frac{(.3)^3}{6} = 0.2955.$$

Let's use the remainder term to estimate the error in the approximation. Since the fourth derivative of $\sin x$ is $\sin x$, and since $\sin x \geq 0$ for $0 \leq x \leq .3$, we have

$$|R_3(.3)| = \frac{\sin(\xi_{.3})}{4!}(.3 - 0)^4,$$

for some number $\xi_{.3}$ between 0 and .3. Using the very rough estimate $\sin(\xi_{.3}) \leq 1$, we arrive at

$$|R_3(.3)| \leq \frac{1}{4!}(.3)^4 = .0003375.$$

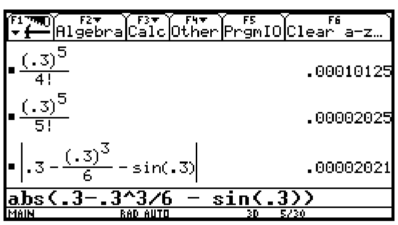
This estimate can be sharpened somewhat by using the fact that $\sin(\xi_{.3}) \leq .3$. (Why is this true?) This gives the estimate

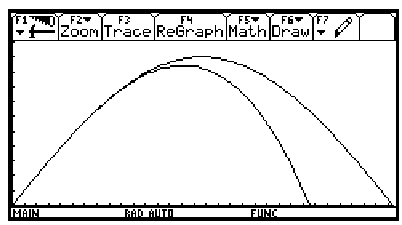
$$|R_3(.3)| \leq \frac{.3}{4!}(.3)^4 = .00010125.$$

A still sharper error estimate can be found by realizing that the fourth degree Taylor polynomial for $\sin x$ about $x = 0$ is the same as the cubic Taylor polynomial about $x = 0$. Therefore we can estimate the error with

$$|R_4(.3)| = \frac{\cos(\xi_{.3})}{5!}(.3 - 0)^5 \leq \frac{(.3)^5}{5!} = .00002025.$$

Finally, there is one more (simpler) method of estimating the error that is available to us. The full series for $\sin(.3)$ is an *alternating series*. Therefore, the error can be estimated by the absolute value of the next nonzero term, which here is $(.3)^5/5!$, the very same estimate that we obtained for $|R_4(.3)|$. The calculations on the left below indicate that this error estimate is quite sharp indeed. The plot on the right below shows the graphs of $\sin x$ and $x - x^3/6$ on the interval $0 \leq x \leq \pi$.





Because of the form of the remainder term, if we want to approximate $f(x)$ over an interval $[a - r, a + r]$, we have the error estimate

$$|R_n(x)| \leq \frac{Mr^{n+1}}{(n+1)!} \quad \text{for } |x - a| \leq r,$$

where

$$M = \max_{[a-r,\, a+r]} \left| f^{(n+1)}(x) \right|.$$

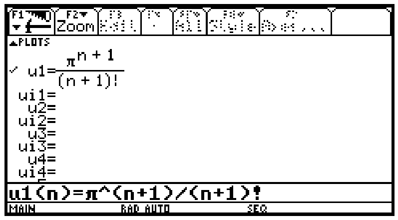
• Example 3. Suppose we wish to find a polynomial approximation to $f(x) = \cos x$ uniformly on the interval $[-\pi, \pi]$ with an error of not more than 0.01. Since f and all of its derivatives have values between -1 and 1, we can take $M = 1$. So, for any x in $[-\pi, \pi]$, we have

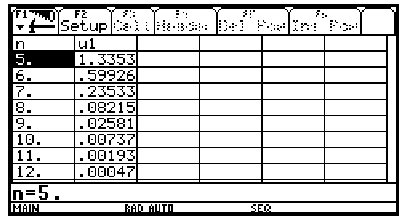
$$|R_n(x)| \leq \frac{\pi^{n+1}}{(n+1)!},$$

and therefore we want to find the least value of n such that

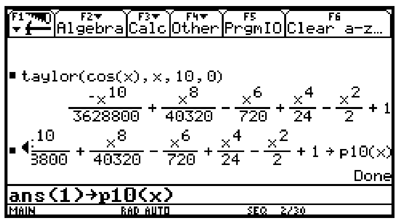
$$\frac{\pi^{n+1}}{(n+1)!} \leq 0.01.$$

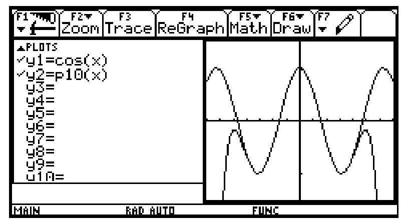
By tabulating the sequence $\pi^{n+1}/(n+1)!$, we see that the desired value is $n = 10$.





Now we plot $\cos x$ together with $p_{10}(x)$ on the interval $[-7,\, 7]$. Notice the closeness of the two graphs between $-\pi$ and π.





Exercises

1. Find the fifth and tenth degree Taylor polynomials for $f(x) = \cos x$ about $x = \frac{\pi}{2}$. Then plot them both along with $\cos x$ on the interval $[-4,\, 7]$.

2. Find the fifth and tenth degree Taylor polynomials for $f(x) = \cos(1 - e^x)$ about $x = 0$ and plot them along with $\cos(1 - e^x)$ on the interval $[-2,\, 3]$.

3. Find the fifth and tenth degree Taylor polynomials for $f(x) = (1 + x^2)^{-1}$ about $x = 0$ and plot them along with $(1 + x^2)^{-1}$ on the interval $[-2,\, 2]$.

4. Find a polynomial approximation to $f(x) = e^x$ on the interval $[-2,\, 2]$ with an error of not more than 0.01. Then plot both the polynomial approximation and e^x on the interval $[-4,\, 4]$.

5. Find a polynomial approximation to $f(x) = \ln x$ on the interval $[1/2,\, 3/2]$ with an error of not more than 0.001. Then plot both the polynomial approximation and $\ln x$ on the interval $[0.01,\, 4]$.

6. Find the third, fourth, and fifth degree Taylor polynomials for $f(x) = x^3 + x^2 + x + 1$ about $x = 0$ and also about $x = 1$. What do you observe?

7. Use the cubic Taylor polynomial for $e^{\sin x}$ to approximate $e^{\sin 0.25}$. Use the bound on the remainder term to estimate the error in the approximation.

8. Use the fourth degree Taylor polynomial for $\sqrt{1 + x^4}$ about $x = 0$ to approximate the integral $\int_0^1 \sqrt{1 + x^4}\, dx$. Estimate the error in the approximation.

9. Use the fourth degree Taylor polynomial for $\cos x$ about $x = 0$ to find an approximate formula for the positive solution of $\cos x = x^2 + k$, in terms of k, for $k < 1$. For what values of k does this give a reasonably accurate solution? *Hint*: *The quadratic formula is your friend.* ☺

10. What other important theorem does Taylor's Theorem become in the case $n = 0$? What does Taylor's Theorem tell us about the linear approximation to a twice differentiable function about $x = a$, i.e., in the case $n = 1$?