“Why Worry About Proving Things?”

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Objective

Scientists, engineers, and mathematicians are concerned not only with equations and formulas, but also with explanations — or proofs — that these equations and formulas are true. In this project we discuss why.

Narrative

Hopefully it goes without saying that it's important to get things right in science, engineering, and mathematics, whether the task involves building a computer chip or sending a rocket to the moon. Statements that are made on the basis of a guess — statements that may at first glance appear to be completely reasonable — under closer examination often turn out to be false. And some statements that are made on the basis of an informal argument often turn out to be false, too. It is because of this that scientists, engineers, and mathematicians are interested in proofs, *rigorous* proofs, of statements. They're interested in convincing arguments that the equations and formulas they use are accurate.

Task

1. a) What is the value of

$$\left(1+\frac{1}{n}\right)^n \tag{1}$$

when n is large? You might guess that it should be close to 1 since, when n is large, $\frac{1}{n}$ is close to 0 so $1+\frac{1}{n}$ is close to 1 and “1 to any power is 1”. But is (1) close to 1 when n is large? Try using the following Maple code to do some experimenting, and see what you get: this code can be used to compute (1) for some large values of n.

```
> # Your name, today´s date
> # Why Worry?
> # Task 1a
> restart;
> for n from 1 to 1000 by 100 do [n,evalf((1+1/n)^n)] end do;
```

The last line of the above code is what is known as a “for loop”. What value does (1) appear to have as n gets very, very large?

b) What value do you think $n^{1/n}$ should have when n is large? Justify your guess. After doing that, write your own Maple code to compute $n^{1/n}$ for large powers of n and see what you get!

2. You might be familiar with the expression

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots.$$

In fact, you might know that the value of this expression is 1. (If you put line segments of lengths 1/2, 1/4, . . . , end-to-end then you would get a line segment whose length approaches 1.) What do you think the value of

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$$

is? Try using the following Maple code to do some more experimenting, and see what you get: this code can be used to compute

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots+\frac{1}{n} \tag{2}$$

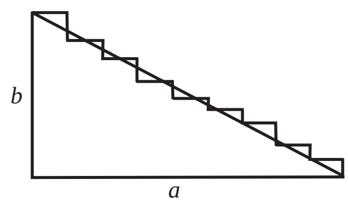
for some very large values of n.

```
> # Task 2
> total := 0.0;
> for n from 2 to 1000000 do total := total + 1/n end do:
> total;
```

In the 2nd line of the above code we declare the number `total` to be 0.0, and in the 3rd line we use a for loop to sequentially add $1/n$ to `total`. The last line of the above code just reports the value `total` has at the end.

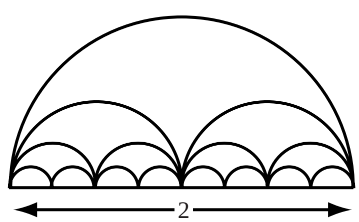
You might be surprised to learn that while the above code may actually make it look like (2) approaches a finite number, we will later be able to prove that (2) gets infinitely large as n gets infinitely large!

3. Suppose you are interested in knowing the length of the hypotenuse of a right triangle. You draw the figure below, and observe that the length c of the hypotenuse is approximately the length of the zig-zag curve; and, indeed, the more steps in the zig-zag curve, the better the zig-zag curve approximates the hypotenuse. Since the sum of the lengths of the horizontal segments of the zig-zag curve is a, and the sum of the lengths of the vertical segments of the zig-zag curve is b no matter how many steps there are, you conclude that $c = a + b$. Or is it?!



The conclusion "$c = a + b$" is, of course, false. (It is in direct conflict with the Pythagorean Theorem: $c^2 = a^2 + b^2$.) So what is wrong with our argument?

4. What is the value of π? Would you believe that $\pi = 2$?! Well, consider the following argument: If the diameter d of the large semicircle is 2, then its circumference is $\pi d/2 = 2\pi/2 = \pi$. (Recall that π is, by definition, the ratio C/d of the circumference C of a circle to its diameter d. Thus $C = \pi d$, and the circumference of a semicircle of diameter d is $\pi d/2$.) Since the diameter d of the next 2 smaller semicircles is 1, the sum of their circumferences is $2 * (\pi d/2) = 2 * (1 * \pi/2) = \pi$. Since the diameter d of the next 4 smaller semicircles is $1/2$, the sum of their circumferences is $4 * (\pi d/2) = 4 * (\frac{1}{2}\pi/2) = 4 * (\pi/4) = \pi$. And so forth. Since the sum of the circumferences of successive sets of semicircles — which is always π — approaches the diameter of the largest semicircle — which is 2 — we conclude that $\pi = 2$. Or is it?!



The statement "$\pi = 2$" is, of course, false. (Actually, $\pi \approx 3.14159265$.) What is wrong with our argument?

At this point, make a hard-copy of your typed input and Maple's responses.

Comments

While the examples presented in this project were meant to be interesting, they were also meant to illustrate the need for proof. The need for proof is more than a phenomenon of mathematics: The examples we discussed in this project came from mathematics since that is the subject we will be studying. Getting away from mathematics, however, consider software engineering; there programmers want to know that the algorithms upon which their code is based yield correct results. (Think of the consequences that arise when computer programs fail.) In engineering, engineers want to know that the structures they build have integrity. (Think of the consequences that arise when buildings, bridges, aircraft, and computers fail.) The search for proof is important in every area of science and engineering.