Factoring Polynomials

Michael Penna, Indiana University – Purdue University, Indianapolis

Objective

To review some tools for factoring polynomials by hand. This project involves computations you should be able to make by hand but which you might want to use Maple to check.

Narrative

A polynomial $p(x)$ of degree n is an algebraic expression of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0, \tag{1}$$

where $a_0, a_1, \ldots, a_{n-1}, a_n$ are constants and $a_n \neq 0$. In this project we review some of the basic tools for factoring polynomials. Knowing how to factor polynomials is important since we will be interested in solving polynomial equations

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 \tag{2}$$

throughout this course, and such equations are easy to solve when $p(x)$ can be factored.

It is easy to solve (2) when $p(x)$ can be factored because of the Zero Factor Theorem.

> *The Zero Factor Theorem*: If $A * B = 0$ then $A = 0$ or $B = 0$.

If we know, for example, that $x^2 - 5x + 6 = (x-2)(x-3)$ then solving the equation $x^2 - 5x + 6 = 0$ is equivalent to solving $(x-2)(x-3) = 0$, and the Zero Factor Theorem allows us to say that the only way this product can be 0 is $x - 2 = 0$ or $x - 3 = 0$; hence $x = 2$ or 3.

One of the most important results dealing with the factoring of polynomials is the Fundamental Theorem of Algebra.

> *The Fundamental Theorem of Algebra*: Any polynomial (1) with complex coefficients can (up to a reordering of factors) be uniquely factored into a product of linear factors of the form $x - r_i$ where each r_i is complex. Further, if the coefficients of (1) are *real* numbers, then the complex roots r_i occur in conjugate pairs, and hence (1) can be factored into a product of linear and irreducible quadratic factors. (A quadratic factor $ax^2 + bx + c$, a, b, and c real, is said to be *irreducible* if it cannot be factored using only real numbers. Equivalently, it is irreducible if the discriminant $b^2 - 4ac < 0$.)

As important as the Fundamental Theorem of Algebra is, in practice it is important to have some practical tools for factoring polynomials by hand. Unfortunately there is no fixed set of rules that will always lead to the complete factorization of a polynomial. There is, however, one strategy that is often useful. It is based on two facts:

1. *If* the coefficients $a_0, a_1, \ldots, a_{n-1}, a_n$ in (2) are *all integers*, and *if* (2) has any *rational* roots, then those roots are of the form
$$\pm \frac{\text{a factor of } a_0}{\text{a factor of } a_n}.$$

2. If r is a root of (2), then $x - r$ is a factor of $p(x)$, so $p(x) = (x - r)q(x)$ for some polynomial $q(x)$ of degree $n - 1$, and to find the remaining roots of (2) it suffices to find the roots of the equation $q(x) = 0$.

By iterating these steps, we can often completely factor (1) and find the roots of (2).

Example: If $p(x) = 6x^3 - 11x^2 - 3x + 2$ then $a_0 = 2$ and $a_3 = 6$, so if $p(x)$ has any rational roots then they are of the form

$$\pm\frac{\text{a factor of } 2}{\text{a factor of } 6}.$$

Since the factors of 2 are 1 and 2, and the factors of 6 are 1, 2, 3, and 6, this implies that if $p(x)$ has any rational roots then they are in the (*finite*) set

$$\pm 1/1 = \pm 1, \quad \pm 1/2 = \pm 1/2, \quad \pm 1/3 = \pm 1/3, \quad \pm 1/6 = \pm 1/6$$

$$\pm 2/1 = \pm 2, \quad \pm 2/2 = \pm 1, \quad \pm 2/3 = \pm 2/3, \quad \pm 2/6 = \pm 1/3$$

or

$$\pm 1, \quad \pm 1/2, \quad \pm 1/3, \quad \pm 1/6, \quad \pm 2, \quad \pm 2/3.$$

Upon checking, we find that $p(1) \neq 0, p(-1) \neq 0$, and $p(1/2) \neq 0$. But $p(-1/2) = 0$! And since this says that $x - (-\frac{1}{2}) = x + \frac{1}{2}$ is a factor of $p(x)$, we divide $p(x)$ by $2x + 1$ to find that

$$\frac{p(x)}{2x+1} = 3x^2 - 7x + 2$$

or

$$p(x) = (2x+1)(3x^2 - 7x + 2).$$

Repeating this process with $q(x) = 3x^2 - 7x + 2$ we find that $q(x) = (3x - 1)(x - 2)$, so that

$$p(x) = (2x+1)(3x^2 - 7x + 2) = (2x+1)(3x-1)(x-2).$$

If we had found that *none* of the possible roots of $q(x)$ actually *was* a root of $q(x)$, then we would know that $q(x)$ has no rational roots, so $p(x) = (2x+1)(3x^2 - 7x + 2)$ would be as far as we could factor $p(x)$ using integer coefficients.

Exercises

Factor the following polynomials completely:

1. $x^2 - 7x + 12$
2. $2x^2 + 5x - 3$
3. $6x^2 - x - 1$
4. $11x^2 - 54x + 63$
5. $x^3 + 2x^2 - x - 2$
6. $4x^3 + 10x^2 - 6x - 18$
7. $3x^3 - 4x^2 - 27x + 36$
8. $9x^3 - 18x^2 - 4x + 8$
9. $x^4 + 3x^3 - x - 3$
10. $3x^4 - x^3 + x - 3$
11. $x^4 - x^3 - 2$
12. $2x^4 + 5x^3 - 5x^2 - 5x + 3$

Note: You can use Maple to check your answers either by factoring or by expanding your factorizations, but you should be able to factor polynomials such as these by hand (since you won't always have Maple to use when you need it).

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