Complex Numbers

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Objective

To briefly review some elementary theory of complex numbers. This project involves computations you can make by hand.

Narrative

A complex number is a number of the form $z = a + bi$ where a and b are real numbers, and $i^2 = -1$. (*Notation*: Some people use j rather than i to denote the number whose square is -1. In the study of electricity, for example, the letter "i" represents current, and electrical engineers often use j rather than i to avoid confusion. Maple uses I rather than i.) Complex numbers arise in the study of quadratic equations such as

$$ax^2 + bx + c = 0.$$

If the discriminant $b^2 - 4ac > 0$ then this equation has 2 unequal real roots

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

if $b^2 - 4ac = 0$ then it has the single double real root $x = -b/2a$, and if $b^2 - 4ac < 0$ then it has no real roots. Once we have defined complex numbers, the above equation has two complex roots when $b^2 - 4ac < 0$. If, for example, our quadratic equation is $x^2 + x + 1 = 0$ then

$$x = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i.$$

The arithmetic of complex numbers is fairly straightforward: it is essentially the same as the arithmetic of real numbers with i being treated as a constant. For example:

addition	$(1 + 3i) + (3 + 2i) = 4 + 5i$
subtraction	$(1 + 3i) - (3 + 2i) = -2 + i$
multiplication	$(1 + 3i)(3 + 2i) = 3 + 11i + 6i^2 = 3 + 11i - 6 = -3 + 11i$
	$(1 + 3i)^2 = (1 + 3i)(1 + 3i) = 1 + 6i + 9i^2 = 1 + 6i - 9 = -8 + 6i$
division	$\frac{1+3i}{3+2i} = \frac{1+3i}{3+2i}\frac{3-2i}{3-2i} = \frac{(1+3i)(3-2i)}{(3+2i)(3-2i)} = \frac{3+7i-6i^2}{9-4i^2} = \frac{9+7i}{13} = \frac{9}{13} + \frac{7}{13}i$

The complex number $\bar{z} = a - bi$ used in the division of a complex number by $z = a + bi$ is called the *complex conjugate* of z. It has the properties that

$$z + \bar{z} = 2a \quad \text{and} \quad z\,\bar{z} = ||z||^2 = a^2 + b^2$$

are real numbers, $||z||$ denoting the magnitude $||z|| = \sqrt{a^2 + b^2}$ of z.

One way in which the arithmetic of complex numbers is special is that the rule $\sqrt{a}\sqrt{b} = \sqrt{ab}$ does not extend to complex numbers: it is true *only* if a and b are positive real numbers: Thus, for example,

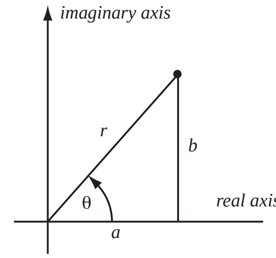
$$\sqrt{-2}\sqrt{-3} \neq \sqrt{(-2)(-3)} = \sqrt{6}.$$

When multiplying the square roots of negative real numbers, roots are extracted *prior* to multiplication. Thus,

$$\sqrt{-2}\sqrt{-3} = (\sqrt{2}i)(\sqrt{3}i) = \sqrt{6}i^2 = -\sqrt{6}.$$

To simplify the computation of products and quotients of complex numbers, it is useful to introduce the trigonometric form of a complex number:

$$z = a + bi = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}}i\right) = r(\cos\theta + i\sin\theta)$$



where $r = \sqrt{a^2+b^2}$ and $\theta = \mathrm{Tan}^{-1}(b/a)$. (See the figure to the right.) Using this form, it follows that if $z_0 = r_0(\cos\theta_0 + i\sin\theta_0)$ and $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ then

$$z_0 z_1 = (r_0(\cos\theta_0 + i\sin\theta_0))(r_1(\cos\theta_1 + i\sin\theta_1)) = (r_0 r_1)(\cos(\theta_0+\theta_1) + i\sin(\theta_0+\theta_1))$$

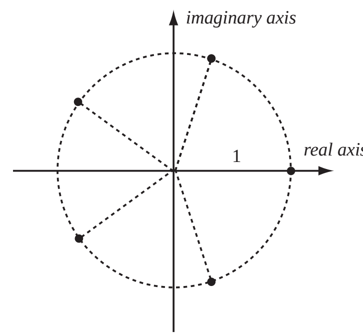
$$\frac{z_0}{z_1} = \frac{r_0(\cos\theta_0 + i\sin\theta_0)}{r_1(\cos\theta_1 + i\sin\theta_1)} = \frac{r_0}{r_1}(\cos(\theta_0-\theta_1) + i\sin(\theta_0-\theta_1))$$

Further, we have DeMoivre's Theorem for computing powers:

$$z^n = (r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta).$$

As far as roots of a complex number are concerned, every complex number has n complex nth roots for every integer n:

$$(r(\cos\theta + i\sin\theta))^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right)$$



for $k = 0, 1, 2, \ldots, n-1$. Geometrically, these roots all lie on a circle in the complex plane whose center is the origin and whose radius is $r^{\frac{1}{n}}$, at equally spaced angles. For example, the equation $z^5 = 1$ has five roots: the five 5th roots of 1 are illustrated in the figure to the right.

Consistent with the theory of Calculus (and power series, in particular) we define the complex number $e^{i\theta}$ by

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

It follows immediately that

$$z = a + bi = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

and further that

$$z_0 z_1 = (r_0 e^{i\theta_0})(r_1 e^{i\theta_1}) = r_0 r_1 e^{i(\theta_0+\theta_1)}$$

$$\frac{z_0}{z_1} = \frac{r_0 e^{i\theta_0}}{r_1 e^{i\theta_1}} = \frac{r_0}{r_1} e^{i(\theta_0-\theta_1)}$$

$$z^{1/n} = r^{1/n} e^{i(\theta/n + 2\pi k/n)}$$

for $k = 0, \ldots, n-1$.

Exercises

Find the roots of:

1. $x^2 + 6x - 1 = 0$

2. $x^2 - 5x + 3 = 0$

3. $x^2 - 4x + 4 = 0$

4. $x^2 + 2x + 1 = 0$

5. $x^2 + 2x + 3 = 0$

6. $x^2 - x - 2 = 0$

7. $2x^2 + 3x + 5 = 0$

8. $3x^2 + 2x + 1 = 0$

Find:

9. $(1+2i)+(2+3i)$
10. $(4+3i)+(2+i)$
11. $(1+2i)-(2+3i)$
12. $(4+3i)-(2+i)$
13. $(1+2i)(2+3i)$
14. $(4+3i)(2+i)$
15. $(1+2i)/(2+3i)$
16. $(4+3i)/(2+i)$
17. $(1+2i)^2$
18. $(4+3i)^2$
19. $(-i)^{19}$
20. i^{53}

Find the magnitude $||z||$ of:

21. $1+2i$
22. $4+3i$
23. $2+3i$
24. $2+i$

Write in the forsm $z = r(\cos\theta + i\sin\theta)$:

25. $1+2i$
26. $4+3i$
27. $2+3i$
28. $2+i$

Find:

29. $(2(\cos 30° + i\sin 30°))\,(3(\cos 15° + i\sin 15°))$
30. $(3(\cos 45° + i\sin 45°))\,(2(\cos 60° + i\sin 60°))$
31. $(2(\cos 30° + i\sin 30°))\,/\,(3(\cos 15° + i\sin 15°))$
32. $(3(\cos 45° + i\sin 45°))\,/\,(2(\cos 60° + i\sin 60°))$
33. $(2(\cos 30° + i\sin 30°))^8$
34. $(3(\cos 45° + i\sin 45°))^{21}$

Find:

35. $1^{1/3}$
36. $1^{1/5}$
37. $i^{1/3}$
38. $i^{1/5}$
39. $(-1)^{1/3}$
40. $(-1)^{1/5}$
41. $(-i)^{1/3}$
42. $(-i)^{1/5}$
43. $(8(\cos 45° + i\sin 45°))^{1/3}$
44. $(243(\cos 45° + i\sin 45°))^{1/5}$
45. $(27(\cos 60° + i\sin 60°))^{1/3}$
46. $(32(\cos 60° + i\sin 60°))^{1/5}$

Comments

1. While every complex number z has n complex nth roots, it is *not* true that every real number x has n real nth roots: For example, $x = -1$ has *no* (or 0) second (or square) real roots, although $z = -1$ has two second (or square) complex roots ... namely $\pm i$.
2. While many of the results of Calculus can be extended from real-valued functions of a real variable to complex-valued functions of a complex variable, all such extensions must be made carefully. In particular, since the logarithmic function ln, the exponential function exp, and the circular trig functions (such as sin, cos, and tan) are defined *only for real numbers*, they must be handled carefully when interpreted as functions of a complex variable.