Linear Interpolation

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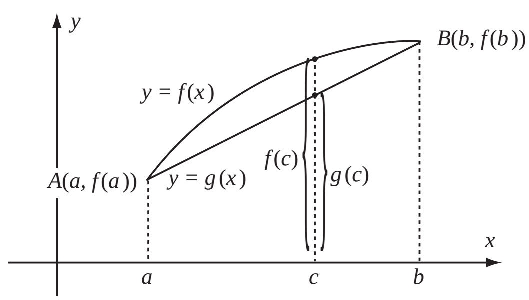
Objective

Some functions can be represented only numerically (by a table of values). Such functions are generally defined for some values of the domain variable x, but not for all such values. The process of linear interpolation allows us to estimate the value of a function f represented numerically, for any value c of x that lies between two values a and b in the domain for which f *is* defined. In this project we discuss linear interpolation, This project involves computations you can make by hand; you may, however, wish to use a hand-calculator or Maple to simplify doing some arithmetic.

Narrative

If you know the values $f(a)$ and $f(b)$ of a function $f = f(x)$ when $x = a$ and $x = b$, and if you know that the values of f over the interval $[a, b]$ do not vary greatly from those of the linear function $g = g(x)$ whose graph passes through the points $A(a, f(a))$ and $B(b, f(b))$ (see the figure below) then you can approximate the value $f(c)$ of $f(x)$ for any $x = c \in [a, b]$ by $g(c)$. The process of doing this is known as *linear interpolation*.



One approach to obtaining the value of $g(c)$ is to observe that since the equation of the line through A and B is

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a),$$

$$g(c) = \frac{f(b) - f(a)}{b - a}(c - a) + f(a). \tag{1}$$

Example 1: Given a function f for which $f(1) = 1.2$ and $f(3) = 4.5$, we can approximate $f(1.5)$ by

$$g(1.5) = \frac{f(3) - f(1)}{3 - 1}(1.5 - 1) + f(1) = \frac{4.5 - 1.2}{3 - 1}(1.5 - 1) + 1.2 = 2.025.$$

Another approach to finding the value of $g(c)$ is completely geometric. Since (see the figure below), $\Delta ACC'$ is similar to $\Delta ABB'$,

$$\frac{CC'}{BB'} = \frac{AC'}{AB'} \quad \text{or} \quad CC' = \frac{AC'}{AB'}BB'.$$

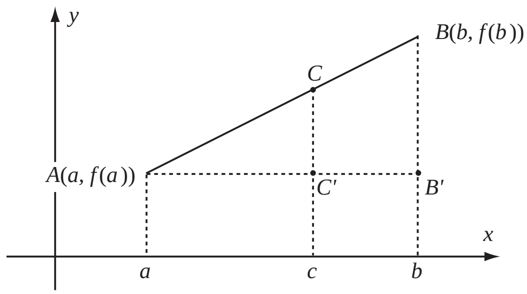
Thus

$$g(c) = f(a) + CC' = f(a) + \frac{AC'}{AB'}BB'.$$

In words, $g(c)$ is — or $f(c)$ is approximately — $f(a)$ plus the same fraction of $f(b) - f(a)$ that $c - a$ is of $b - a$. Since

$$g(c) = \frac{f(b) - f(a)}{b - a}(c - a) + f(a) = f(a) + \frac{c - a}{b - a}(f(b) - f(a))$$

this is exactly the same value we obtained by (1).



Example 2: Repeating the above example, observe that since the ratio of $c - a = 1.5 - 1 = 0.5$ to $b - a = 3 - 1 = 2$ is $0.5/2 = 0.25$, 1.5 is 1/4th of the way from 1 to 3. Hence $g(1.5)$ is 1/4th of the way from $f(1)$ to $f(3)$:

$$g(1.5) = f(1) + 0.25 * (f(3) - f(1)) = 1.2 + 0.25 * (4.5 - 1.2) = 2.025.$$

Exercises

Assuming that:

1. $\sin 23° = 0.3907$ and $\sin 24° = 0.4067$, approximate $\sin 23.6°$.

2. $\sin 100° = 0.9848$ and $\sin 101° = 0.9816$, approximate $\sin 100.4°$.

3. $\sin 210° = -0.5000$ and $\sin 211° = -0.5150$, approximate $\sin 210.1°$.

4. $\log_{10} 3 = 0.4771$ and $\log_{10} 4 = 0.6020$, approximate $\log_{10} 3.2$.

5. $\log_{10} 0.5 = -0.3010$ and $\log_{10} 0.6 = -0.2218$, approximate $\log_{10} 0.58$.

6. $\log_{10} 1.2 = 0.0791$ and $\log_{10} 1.3 = 0.1139$, approximate $\log_{10} 1.24$.