Guessing Limits Numerically

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Objective

To guess the limit of a function at a point numerically.

Narrative

Prior to having the theorems on limits at our disposal, there are two major issues surrounding the limit of a function at a point: The first is guessing what the limit is, if it even exists; this issue can often be approached either graphically or numerically. The second issue involves proving that the guess you made is correct; this issue involves using the formal definition of limit.

In this project we address the issue of guessing limits numerically. In this project we also illustrate how to perform repeated computations efficiently in Maple using a "do loop".

Tasks

1. Type the command lines in the left-hand column below into Maple in the order in which they are listed. These commands will help you estimate $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$ numerically, if it exists. (*Note*: It's OK to type either of the loops "`for n from 1 to 6 do ...  end do:`" on one line.)

`> # Your name, today´s date`	
`> # Guessing Limits Numerically`	
`> restart;`	Clear Maple's memory.
`> f := x -> (1-cos(x))/x^2;`	Let $f(x) = (1-\cos x)/x^2$.
`> plot(f(x),x=-1..1);`	Plot the graph of f.
`> f(0);`	What is $f(0)$?
`> for n from 1 to 6`	Let's look at some values of $f(x)$ for $x < 0$.
`do`	This is the beginning of a "do loop".
`x := -1/2^n:`	Let $x = -\frac{1}{2^n}$.
`print(evalf(x), evalf(f(x)));`	Print the values of x and $f(x)$.
`end do:`	This is the end of the "do loop".
`> for n from 1 to 6`	Now let's look at some values of $f(x)$ for $x > 0$.
`do`	
`x := 1/2^n:`	
`print(evalf(x), evalf(f(x)));`	
`end do:`	

At this time make a hard-copy of your typed input and Maple's responses. Then:

2. On the graphic you created in Task 1, plot the points on the graph of f whose y values you computed in the two "do loops". (The easiest way to do this is to plot x values along the x-axis and then draw lines parallel to the y-axis through these points, until they meet the graph of f.)

3. On the basis of this data, do you think $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$ exists? If so, what do you think it is (to 4 decimal places of accuracy)? Justify your answer.

Your lab report will be a hard copy of your typed input and Maple's responses (both text and hand-labeled graphics).

Comments

1. You can *guess* whether or not $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$ exists, and if it does exist, what its value is, on the basis of numerical "evidence" (as we did in this project), but you *cannot say for sure* that you're correct: you can never perform more than a finite number of computations, and however close x is to 0, you may *miss* some critical behavior of $f(x) = \dfrac{1-\cos x}{x^2}$ that might affect your guess. It is because of this that we *must* turn to the formal concept of the limit.

2. Different rates of convergence can be achieved by replacing `1/2^n` by `1/n^2` (this produces a slower rate of convergence) or `1/n^n` (this produces a faster rate of convergence).

3. The physical limitations of your computer may limit the accuracy of your computations.

4. Maple has a built in command "`limit(f(x),x=a)`" that allows you to compute (some) limits automatically. (Variations on this include "`limit(f(x),x=infinity)`" and "`limit(f(x),x=-infinity)`" for computing limits at $\pm\infty$, and "`limit(f(x),x=a,left)`" and "`limit(f(x),x=a,right)`" for computing left- and right-hand limits.) Since we are interested not just in what limits are, but how they are computed, we intentionally avoided using this command in this project.

5. At the end of the `do` loops in the above code, Maple will think that $n = 6$ and $x = a \pm 1/2^n$. (You can check this by entering the commands "`n;`" and "`x;`" after each loop.) This is important to know since if, subsequent to the appropriate `do` loop, you wanted to reuse n or x as a variable then you would have to redefine it as a variable using the command "`n := ´n´`" or the command "`x := ´x´`".