The Definition of Limit

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Objective

To investigate the precise definition of limit.

Narrative

To prove that what we *guess* to be the limit of $f(x)$ at $x = a$ *really is* the limit of $f(x)$ at $x = a$, we must verify that for each real number $\epsilon > 0$, there is a real number $\delta > 0$ such that the values of $f(x)$ for all x in the interval $(a - \delta, a + \delta)$ — except possibly at $x = a$ itself — lie between $L - \epsilon$ and $L + \epsilon$. In this project we investigate the graphical implications of this condition.

Task

1. Type the command lines below into Maple in the order in which they are listed. These commands are concerned with $\lim_{x \to 1}(x^3 - x^2 + 2x - 2)/(x - 1)$.

```
> # Your name, today´s date
> # The Definition of Limit
> restart;
> f := x -> (x^3-x^2+2*x-2)/(x-1);
> a := 1.0;
> L := limit(f(x),x=a);
> e := 0.5;
> d := 1.0;
> plot(f(x),x=a-d..a+d,y=L-e..L+e,axes=boxed,scaling=constrained);
```

2. Using trial-and-error, find (to 2 decimal places of accuracy) the largest value of `d` for which the values of `f(x)` for all `x` in the interval `a-d..a+d` — except possibly at $x = a$ — lie between `L-e` and `L+e`. You can do this by copying the last two lines of the code above, and then repeatedly modifying the value of `d` and using the graphic to check whether the condition is true.

At this time make a hard-copy of your typed input and Maple's responses. Then:

3. By hand, draw the lines $x = a$ and $y = L$, and label these lines as well as the lines whose equations are $x = a \pm \delta$ and $y = L \pm \epsilon$, in each of the (two) graphics you created in Task 1.

Your lab report will be a hard copy of your typed input and Maple's responses for the *initial* case (when `d = 1.0`) and the *final* case (when `d` is "small enough"), *not* the intermediate trial-and-error input/output!

Comments

In this task you are *not* actually proving that $L = \lim_{x \to a} f(x)$. On one hand, you are just verifying that an appropriate `d` exists for *one* given `e`: to verify that $L = \lim_{x \to a} f(x)$, you would have to do this *for every* `e`, not just one. On the other hand, since Maple draws the graphs of functions by "connecting-the-dots", some significant behavior could occur *between* the dots that is not revealed by Maple.