The Intermediate Value Theorem and Graphing

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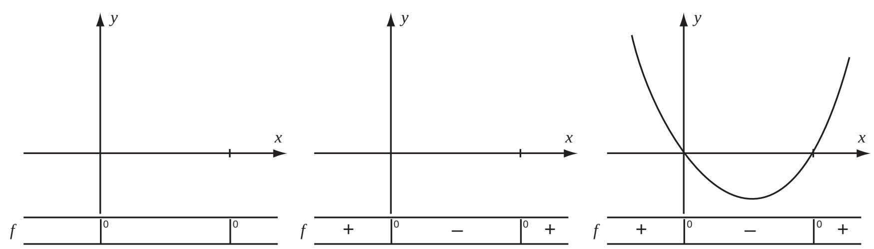
Objective

One of the consequences of the Intermediate Value Theorem is a method for drawing a qualitatively accurate graph of a rational function $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials. In this project we discuss this method.

Narrative

For the sake of simplicity, we proceed by example. To graph a polynomial function such as $f(x) = (x^4 - 4x^3)/10$, we first find the zeros of f (the values of x for which $f(x) = 0$). Since $f(x) = (x^4 - 4x^3)/10 = x^3(x-4)/10$, the zeros of f are 0 and 4. Next we draw some vertical bars in a recording strip below a blank graph at 0 and 4 (as illustrated below). The zeros 0 and 4 divide the domain of f into 3 subintervals, and f is either positive or negative over each of these subintervals. (To justify the assertion that, "f is either positive or negative over each of these subintervals", consider, for example, the interval $(-\infty, 0)$: if f were positive at one point and negative at another point in this subinterval then — since f is a polynomial and all polynomials are continuous — there would, according to the Intermediate Value Theorem, be a zero of f between these points; but the only zeroes of f are at 0 and 4, so this is impossible.)



To determine the sign of f over $(-\infty, 0)$, we thus simply determine the sign of f at one point in $(-\infty, 0)$: since $f(-1) > 0$, for example, f is positive on $(-\infty, 0)$. Similarly we find f is negative on $(0, 4)$, and positive on $(4, \infty)$. We add this information to our recording strip, and using this information we sketch the graph of f.

If f were a rational function instead of a polynomial function then we would not only find and use the points at which f (or the numerator of f) is 0, but we would also find and use the points at which f is undefined (or the denominator of f is 0), as well as $\lim_{x\to\pm\infty} f(x)$: the points at which f is undefined and $\lim_{x\to\pm\infty} f(x)$ contain information about the vertical and horizontal asymptotes of f, respectively.

Tasks

1. Type the command lines below into Maple in the order in which they are listed. They draw a blank coordinate system and recording strip; they also define the function $f(x) = x^4 + x^3 - 6x^2$ and find its zeros. (Later you will be asked to use your output to graph f.)

```
> # Your name, today´s date
> # The Intermediate Value Theorem and Graphing
> restart;
> with(plots):
> # Task 1
> plot0 := plot({-6,-5,0,[[0,-5],[0,6]]},x=-6..6,y=-6..6,tickmarks=[0,0],axes=none):
> plot1 := textplot([-6,-5.5,f]):
> display({plot0,plot1});
> f := x -> x^4+x^3-6*x^2;
> solve(f(x)=0,x);
```

2. Continue by typing the command lines below into Maple in the order in which they are listed. They just repeat Task 1 with a rational function g that has both vertical asymptotes and a horizontal asymptote; thus we find not only its zeros (where g's numerator is 0), but also where g is undefined (where g's denominator is 0), and $\lim_{x\to\pm\infty} g(x)$.

```
> # Task 2
> display({plot0,plot1});
> g := x -> 4*x/(1-x^2);
> solve(numer(g(x))=0,x);
> solve(denom(g(x))=0,x);
> limit(g(x),x=infinity);
> limit(g(x),x=-infinity);
```

At this time make a hard-copy of your typed input and Maple's responses. Then:

3. Use the data you produced in Tasks 1 and 2 to sketch the graphs f and g by hand using the technique discussed in the Narrative. Be sure to add and label tick marks on the x-axis.

Your lab report will be a hard copy of your typed input and Maple's responses (both text and hand-labeled graphics).

Comments

When we say that the graph of a function is "qualitatively accurate" (as we did in the Objective for this project) we mean that the graph is correct insofar as the intercepts and asymptotes are correct, and where the graph of f lies above or below the x-axis and the horizontal asymptotes are correct, but that the actual shape of the curve may not be exactly correct beyond this. (A statement is "quantitatively correct" if it is numerically correct; a statement is "qualitatively correct" if it is correct in a general sense, but it is not correct relative to some specific quantities or numbers.) We will expand on the qualitative statements that can be made about the graph of a function later in this course.