Implicit Functions and Implicit Differentiation

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Objective

To illustrate the value of using the implicit description of functions.

Narrative

A function $y = f(x)$ can be described either explicitly — in the form a "recipe" for computing y given x — or implicitly — as the solution to an equation in x and y. While all functions that can be described explicitly can be described implicitly, functions that can be described implicitly cannot always be described explicitly. In this project we illustrate that even if a function that can be described implicitly can be described explicitly, implicit methods — for operations such as differentiation — are sometimes preferable to explicit methods.

Task

1. Type the command lines below into Maple in the order in which they are listed. They instruct Maple to find the derivative of a function $y = f(x)$ defined implicitly by the equation $x^3 + y^3 = 2xy$ *implicitly*, and to *implicitly* plot the graph of this equation. Observe that in this code we are careful in identifying $y = y(x)$ as a function of x rather than just as a variable (independent of x).

```
> # Your name, today´s date
> # Implicit Differentiation
> # Task 1
> restart;
> with(plots):
> eqn0 := x^3+y(x)^3=2*x*y(x);                          Let eqn0 be the equation x^3 + y^3 = 2xy.
> eqn1 := diff(lhs(eqn0),x) = diff(rhs(eqn0),x);        Equate the derivative of the left-hand
                                                           side of eqn0 to the derivative of the
                                                           right-hand side of eqn0.
> Diff(y(x),x) = solve(eqn1,diff(y(x),x));              Solve eqn1 for y'.
> implicitplot(eqn0,x=-2..2,y=-2..2,numpoints=1000, scaling=constrained);
                                                        Graph the equation x^3 + y^3 = xy implicitly.
```

The way Maple plots the graph of the equation $x^3 + y^3 = 2xy$ implicitly is by evaluating $x^3 + y^3 - 2xy$ at each point in a (uniformly spaced) `numpoints`×`numpoints` grid of points in the square $[-1, 1] \times [-1, 1]$: if $|(x^3 + y^3) - 2xy|$ is "adequately small" then Maple plots a point; otherwise, it does not. The way the size of `numpoints` affects the coarseness of the graph can be seen in the appearance of the graph near the origin: the larger `numpoints`, the finer the graph.

2. Continue by typing the following command lines into Maple. They instruct Maple to solve the equation $x^3 + y^3 = 2xy$ for y *explicitly* in terms of x, and to *explicitly* find the derivative of the resulting function.

```
> # Task 2
> y := ´y´;                                 Reestablish y as a variable.
> assume(y,real);                           Assume that y is a real variable.
> solve(x^3+y^3=2*x*y,y);                   Solve x^3 + y^3 = 2xy for y. Observe that
                                               there's more than one solution.
> %[1];                                     Pick the first solution, ...
> Diff(y(x),x) = diff(%,x);                    and differentiate it with respect to x.
```

As noted above, multiple functions are defined by the equation $x^3 + y^3 = 2xy$ (the graph of this equation produced in Task 1 does *not* pass the vertical line test), and hence in computing the derivative of a function defined explicitly by this equation (in Task 2) we must make a choice as to which *branch* we are considering. This is what the "`%[1]`" command does.

Also observe that there is a tilde (~) following the `y` after the `solve` command. This tilde is present to remind us that an assumption has been made about the variable `y`.

Finally observe how much simpler implicit computations are than explicit computations.

At this time make a hard-copy of your typed input and Maple's responses. Then:

3. By hand,

a) find the slope dy/dx at $P(1,1)$,

b) find and plot (on the graphic you created) the points at which $dy/dx = 0$,

c) find and plot (on the graphic you created) the points at which dy/dx does not exist, and

d) find d^2y/dx^2, and express it as a function of x and y.

Your lab report will be a hard copy of your typed input and Maple's responses (both text and hand-labeled graphics), and your written responses.