

Finding Relative Extrema on an Open Interval: The Second Derivative Test

Michael Penna, Indiana University – Purdue University, Indianapolis

Objective

To illustrate how to find and classify the relative extrema of a function on an open interval using the Second Derivative Test.

Narrative

Recall that to find the maximum and minimum values of any function $f(x)$ on an open interval using the Second Derivative Test we:

1. find the critical numbers of f (the values of x for which $f'(x) = 0$ and $f'(x)$ does not exist),
2. compute the sign of the second derivative f'' of f at each critical number c : if
 - (a) $f''(c) > 0$ then $f(c)$ is a relative minimum,
 - (b) $f''(c) < 0$ then $f(c)$ is a relative maximum,
 - (c) $f''(c) = 0$ then the test fails, and we must resort to the First Derivative Test: we compute the sign of the first derivative f' of f at points x_0 and x_1 just to the left and right of the critical number c ($x_0 < c < x_1$); if
 - i. $f'(x_0) > 0$ and $f'(x_1) < 0$ then $f(c)$ is a relative maximum,
 - ii. $f'(x_0) < 0$ and $f'(x_1) > 0$ then $f(c)$ is a relative minimum,
 - iii. $f'(x_0) > 0$ and $f'(x_1) > 0$, or $f'(x_0) < 0$ and $f'(x_1) < 0$, then $f(c)$ is neither a relative maximum nor a relative minimum. (The point $P(c, f(c))$ is a saddle point.)

In this project we illustrate this process as we find the relative maximum and minimum values of $f(x) = 4x^2/(x^4 + 8)$.

Tasks

1. a) Type the command lines in the left-hand column below into Maple in the order in which they are listed. The effect of each command is described in the right-hand column for your reference.

> # Your name, today's date	
> # The Second Derivative Test	
> restart;	Clear Maple's memory.
> Digits := 5;	
> f := x -> 4*x^2/(x^4+8);	Let $f(x) = 4x^2/(x^4 + 8)$.
> f1 := D(f);	Let f1 denote the first derivative f' of f .
> f1exp := simplify(D(f)(x));	Let f1exp denote the simplified expression $f'(x)$.
> fsolve(numer(f1exp)=0,x);	Find where the numerator of $f'(x)$ is 0.
> fsolve(denom(f1exp)=0,x);	Find where the denominator of $f'(x)$ is 0.
> f2 := D(f1);	Let f2 denote the second derivative f'' of f .

b) Use Maple to evaluate f'' (or **f2**) at each critical number c . On the basis of your computations, write Maple comments stating whether $f(c)$ is a relative maximum or relative minimum, or whether $P(c, f(c))$ is a saddle point, for each critical number c . (Your comment might look like, “# is a relative maximum” for example.) If $f''(c) = 0$, use the First Derivative Test at c to determine whether $f(c)$ is a relative maximum or a relative minimum, or whether $P(c, f(c))$ is a saddle point.

c) Type the following command line into Maple, substituting for **a**, **b**, **c** and **d** numbers that are large enough to enclose all the critical numbers of f and the graph of f over $[a, b]$.

> plot(f(x),x=a..b,y=c..d);

Plot the graph of f over the interval $[a, b]$.

At this time make a hard-copy of your typed input and Maple's responses. Then:

2. By hand, on the graphic you produced in Task 1, label the graph of f , plot the critical numbers of f along the x -axis, and plot and label the points $P(c, f(c))$, c a critical number of f . Remembering that the relative extrema of f are y -values, estimate the relative extrema of f by eye and for each relative extremum write a sentence under your graphic of the form, " f has a relative maximum of $_$ when $x = _$ " or " f has a relative minimum of $_$ when $x = _$ ".

Your lab report will be a hard-copy of your typed input and Maple's responses (both text and hand-labeled graphics).