Finding Relative Extrema on an Open Interval: The Second Derivative Test

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Objective

To illustrate how to find and classify the relative extrema of a function on an open interval using the Second Derivative Test.

Narrative

Recall that to find the maximum and minimum values of any function $f(x)$ on an open interval using the Second Derivative Test we:

1. find the critical numbers of f (the values of x for which $f'(x) = 0$ and $f'(x)$ does not exist),
2. compute the sign of the second derivative f'' of f at each critical number c: if
   (a) $f''(c) > 0$ then $f(c)$ is a relative minimum,
   (b) $f''(c) < 0$ then $f(c)$ is a relative maximum,
   (c) $f''(c) = 0$ then the test fails, and we must resort to the First Derivative Test: we compute the sign of the first derivative f' of f at points x_0 and x_1 just to the left and right of the critical number c ($x_0 < c < x_1$); if
      i. $f'(x_0) > 0$ and $f'(x_1) < 0$ then $f(c)$ is a relative maximum,
      ii. $f'(x_0) < 0$ and $f'(x_1) > 0$ then $f(c)$ is a relative minimum,
      iii. $f'(x_0) > 0$ and $f'(x_1) > 0$, or $f'(x_0) < 0$ and $f'(x_1) < 0$, then $f(c)$ is a neither a relative maximum nor a relative minimum. (The point $P(c, f(c))$ is a saddle point.)

In this project we illustrate this process as we find the relative maximum and minimum values of $f(x) = x(1 - x^2)^{2/3}$.

Tasks

1. a) Type the command lines in the left-hand column below into Maple in the order in which they are listed. The effect of each command is described in the right-hand column for your reference.

`> # Your name, today´s date`	
`> # The Second Derivative Test`	
`> restart;`	Clear Maple's memory.
`> Digits := 5;`	
`> f := x -> x*surd(1-x^2,3)^2;`	Let $f(x) = x(1 - x^2)^{2/3}$.
`> f1 := D(f);`	Let `f1` denote the first derivative f' of f.
`> f1exp := simplify(D(f)(x));`	Let `f1exp` denote the simplified expression $f'(x)$.
`> evalf(solve(numer(f1exp)=0,x));`	Find where the numerator of $f'(x)$ is 0.
`> evalf(solve(denom(f1exp)=0,x));`	Find where the denominator of $f'(x)$ is 0.
`> f2 := D(f1);`	Let `f2` denote the second derivative f'' of f.

b) Use Maple to evaluate f'' (or `f2`) at each critical number c. On the basis of your computations, write Maple comments stating whether $f(c)$ is a relative maximun or relative minimum, or whether $P(c, f(c))$ is a saddle point, for each critical number c. (Your comment might look like, "`# __ is a relative maximum`" for example.) If $f''(c) = 0$, use the First Derivative Test at c to determine whether $f(c)$ is a relative maximum or a relative minimum, or whether $P(c, f(c))$ is a saddle point.

c) Type the following command line into Maple, substituting for `a`, `b`, `c` and `d` numbers that are large enough to enclose all the critical numbers of f and the graph of f over $[a, b]$.

```
> plot(f(x),x=a..b,y=c..d);
```
Plot the graph of f over the interval $[a, b]$.

At this time make a hard-copy of your typed input and Maple's responses. Then:

2. By hand, on the graphic you produced in Task 1, label the graph of f, plot the critical numbers of f along the x-axis, and plot and label the points $P(c, f(c))$, c a critical number of f. Remembering that the relative extrema of f are y-values, estimate the relative extrema of f by eye and for each relative extremum write a sentence under your graphic of the form, "f has a relative maximum of _ when $x =$ _" or "f has a relative minimum of _ when $x =$ _".

Your lab report will be a hard-copy of your typed input and Maple's responses (both text and hand-labeled graphics).