Iteration and Chaos

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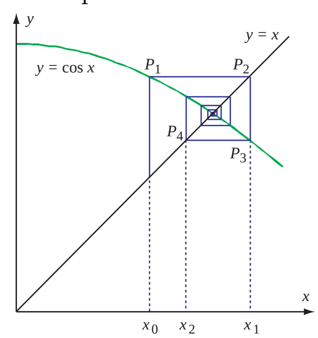
Objective

To illustrate how chaotic behavior can arise in iteration.

Narrative

The Iteration Method and Newton's Method are examples of first-order iterative processes. In general, a first-order iterative process involves starting with a quantity x_0, applying some function f to x_0 to arrive at a quantity $x_1 = f(x_0)$, applying f to x_1 to arrive at $x_2 = f(x_1)$, and so forth. The numbers $x_0, x_1, x_2, \ldots$ form an *infinite sequence*, and we will have much more to say about infinite sequences later.

For now, though, we note that one way to visualize a first-order iterative process is with *cobweb diagrams*. If, for example, $x_0 = 0.5$ and $x_n = f(x_{n-1})$ for $n = 1, 2, \ldots$, then we can visualize the sequence $x_0, x_1, x_2, \ldots$ as follows (see the figure to the right):



1. Since $x_1 = f(x_0)$, the vertical line through x_0 meets the graph of $y = \cos x$ at the point $P_1(x_0, x_1)$ whose y-coordinate is $x_1 = f(x_0)$.
2. To locate $x_1 = f(x_0)$ on the x-axis, we draw a horizontal line through P_1 and project the point P_2 at which this line meets the line $y = x$ onto the x-axis.
3. Since $x_2 = f(x_1)$, the vertical line through x_1 meets the graph of $y = \cos x$ at the point $P_3(x_1, x_2)$ whose y-coordinate is $x_2 = f(x_1)$.
4. To locate $x_2 = f(x_1)$ on the x-axis, we draw a horizontal line through P_3 and project the point P_4 at which this line meets the line $y = x$ onto the x-axis.

We repeat the steps described above over and over again. Can you see the geometric pattern? Starting from x_0, we go vertically to the curve, horizontally to the line, vertically to the curve, horizontally to the line, vertically to the curve, etc. The "limit" $P_\infty(x_\infty, x_\infty)$ of the points P_n is the point of intersection of the graphs of $y = x$ and $y = \cos x$. (Cobweb diagrams get their name from their cobweb-like appearance.)

Tasks

1. a) Type the following command lines into Maple in the order in which they are listed. They generate some terms in the sequence discussed in the Narrative.

```
> # Your name, today´s date
> # Iteration and Chaos
> # Task 1a
> restart:
> f := x -> cos(x);
> x[0] := 0.5;
> for n from 1 to 20 do x[n] := f(x[n-1]); end do;
```

b) Continue by typing the following command lines into Maple in the order in which they are listed. They create the graphic discussed in the Narrative.

```
> # Task 1b
> with(plots):
> mylist := [[x[0],x[0]],[x[0],x[1]]],[[x[0],x[1]],[x[1],x[1]]]:
> for n from 1 to 10 do
    mylist := mylist, [[x[n],x[n]],[x[n],x[n+1]]],[[x[n],x[n+1]],[x[n+1],x[n+1]]]:
> end do:
> A := plot({x,f(x)},x=0..1,thickness=2):
> B := plot([mylist]):
> display({A,B},scaling=constrained);
```

One of the reasons people study cobweb diagrams is in the hope that they might provide insight into the behavior of iterative processes: while a great deal is known about iterative processes, a great deal is also *unknown* about them. To illustrate some of the the subtle behavior of iterative processes, consider the first-order iterative process

$$x_0 = a, \quad x_{n+1} = f(x_n) = kx_n(1 - x_n), \quad n = 0, 1, 2, \ldots. \qquad (*)$$

If $k = 4$ then for some values of $a \in [0, 1]$, x_n converges, for some it repeats, and for others it wanders aimlessly over the interval $[0, 1]$. This type of behavior is known as *chaos*. The study of chaos is important since chaotic behavior arises not only in many areas of science and engineering, but also in areas as diverse as medicine — for example, in the study of dynamic blood diseases — and economics.

2. a) Assuming $k = 4$, find 3 values of $a \in [0, 1]$:

1. one for which x_n converges,
2. one for which x_n repeats, and
3. one for which x_n wanders aimlessly over the interval $[0, 1]$.

Illustrate your results graphically.

b) Assuming $k = 3.839$, choose any number $a \in [0, 1]$, and show that x_n eventually settles down to cyclically repeating — or *orbiting* — the three numbers 0.149888, 0.489172, and 0.959299. This type of behavior is known as *periodicity*.

3. What can be said about the behavior of (*) if $f(x) = x^2 - 1$: what happens if $a = 0$ or 1? what happens if a takes on any other value?

4. What can be said about the behavior of (*) if $f(x) = x^3$?

5. What can be said about the iteration

$$x_{n+1} = x_n - \frac{x_n^3 - 3x_n^2 + 1}{3x_n^2 - 6x_n},$$

which arises in trying to apply Newton's Method to $x^3 - 3x^2 + 1$, when $x_0 = 1.648$, when $x_0 = 1.649$, and when $x_0 = 1.650$?