Linear Approximation and Differentials

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Objective

To illustrate the connection between the linear approximation of a function and its differential.

Narrative

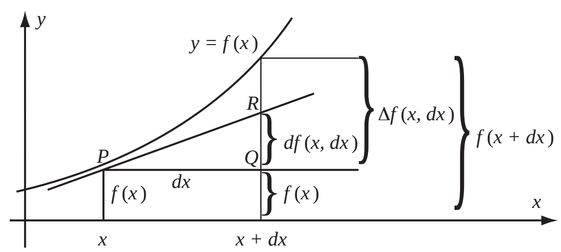
Linear approximations and differentials are important since they allow us to estimate the values of functions that are otherwise difficult or even impossible to evaluate. In this project we illustrate this by using differentials to approximate $\sqrt{1.5}$.

Recall that the differential df of the function $f = f(x)$ is the function of two variables that associates to the pair of numbers x and dx the vertical distance from Q to R illustrated in the figure below. Thus

$$\frac{df}{dx} = \frac{df(x, dx)}{dx} = f'(x) \quad \text{or} \quad df = df(x, dx) = f'(x)dx$$

and

$$f(x + dx) = f(x) + \Delta f(x, dx) \approx f(x) + df(x, dx) = f(x) + f'(x) * dx.$$



Task

1. Type the command lines in the left-hand column below into Maple in the order in which they are listed.

`> # Your name, today´s date`	
`> # Linear Approximation and Differentials`	
`> restart;`	Clear Maple's memory.
`> f := x -> sqrt(x);`	Let $f(x) = \sqrt{x}$.
`> f(1);`	Evaluate $f(x)$ when $x = 1$.
`> df := (x,dx) -> (D(f)(x))*dx;`	Let df be the differential of f.
`> df(1,0.5);`	Evaluate $df(x, dx)$ when $x = 1$ and $dx = 0.5$.
`> f(1)+df(1,0.5);`	The differential approximation to $f(1.5)$.
`> f(1.5);`	The actual value of $f(1.5)$.
`> plot({f(x),f(1)+D(f)(1)*(x-1)},x=0..2,y=0..1.5);`	
	Plot the graph of $f(x)$ near $x = 1$, and the tangent line to the graph of f at $P(1, f(1))$.

At this time make a hard-copy of your typed input and Maple's responses. Then, on the graphic you created in Task 1:

2. draw by hand the horizontal line whose equation is $y = f(1)$, and the vertical lines whose equations are $x = 1.0$ and $x = 1.5$, and label each with its equation, and

3. clearly identify and label by hand the segments whose lengths are $\Delta f(1, 0.5)$ and $df(1, 0.5)$.

4. To four decimal places of accuracy, state the error in computing $\sqrt{1.5}$ by using differentials.

Your lab report will be a hard copy of your typed input and Maple's responses (both text and hand-labeled graphics), and your approximation.

Comments

The example used in this project is, to some extent, extreme: the relative error is large. We used this example since the associated graphic is easy to work with. In many cases, differentials do a much better job of approximating; "unfortunately" in these cases the associated graphic does not lend itself to visualization.