# Integration: Riemann Sums

*Michael Penna, Indiana University – Purdue University, Indianapolis*

## *Objective*

To investigate the approximation of area using Riemann sums.

## *Narrative*

Recall that if $f(x) \geq 0$ for $x \in [a, b]$ then we may approximate the area of the region bounded by the graph of $f$, $x = a$, $x = b$, and the $y$-axis by:

1. subdividing $[a, b]$ into $n$ subintervals, each of length $\Delta x = (b - a)/n$,
2. choosing an "sample point" $x_i^*$ in each subinterval, and
3. computing the sum $\Sigma_{i=1}^{n} f(x_i^*)\Delta x$.

In this project we will use Maple to illustrate these computations.

## *Task*

**1**. Type the command lines in the left-hand column below into Maple in the order in which they are listed. These commands are aimed at approximating the area under the graph of $f(x) = x^2 + x$ from $x = 1$ to $x = 3$ using left-hand endpoints. The effect of each command is described in the right-hand column for your reference.

| | |
|---|---|
| `> # Your name, today´s date` | |
| `> # Integration:  Riemann Sums` | |
| `> # Task 1` | |
| `> restart;` | Clear Maple's memory. |
| `> f := x -> x^2+x;` | Let $f(x) = x^2 + x$. |
| `> a := 1; b := 3;` | Let $a = 1$ and $b = 3$. |
| `> for n from 1 to 100 by 20 do` | For each 20th integer $n$ between 1 and 100, ... |
| `delta_x := (b-a)/n:` | let `delta_x` denote the length of each subinterval. |
| `LHSum := 0.0:` | Starting off with an area `LHSum` of 0.0, ... |
| `for i from 1 to n do` | for each subinterval ... |
| `x_i := a+(i-1)*delta_x;` | let `x_i` be the left-hand endpoint, and ... |
| `LHSum := LHSum + f(x_i)*delta_x:` | add to `LHSum` the area of the $i$th rectangle. |
| `end do:` | |
| `evalf(LHSum);` | Report the value of `LHSum` when done. |
| `end do;` | |

Observe that we use `x_i` in the above code to denote the $x$-coordinate of the $i$th "sample point" $(x_i^*, f(x_i^*))$, $i = 1, \ldots, n$, (instead of $x_i^*$, since Maple interprets $*$ as multiplication), and that in computing:

- the left-hand sum `LHSum` is $x_i^* = a + (i - 1)dx$,
- the right-hand sum `RHSum` is $x_i^* = a + i\ dx$, and
- the midpoint-hand sum `MPSum` is $x_i^* = a + \dfrac{(i-1)+i}{2}dx = a + \dfrac{2i-1}{2}dx$

($x_i^*$ for `MPSum` being the average of the $x_i^*$'s for `LHSum` and `RHSum`).

**2**. a) Enter the appropriate code for approximating the area under the graph of $f(x) = x^2 + x$ from $x = 1$ to $x = 3$ "`for n from 1 to 100 by 20`" using right-hand endpoints.

b) Enter the appropriate code for approximating the area under the graph of $f(x) = x^2 + x$ from $x = 1$ to $x = 3$ "`for n from 1 to 100 by 20`" using midpoints.

**3**. Continue by typing the command line in the left-hand column below into Maple, using for `c` and `d` numbers that are large enough to enclose the region bounded by the $x$-axis, the graph of $f(x)$, and the lines $x = a$ and $x = b$.

```
> # Task 3
> plot(f(x),x=a..b,y=c..d); %; %;
```
Draw *three* copies of the graph of $f$.

(Later, after you have made a hard copy of your typed input and Maple's responses, you will be asked to draw the rectangles and plot the sample points $(x_i^*, f(x_i^*))$ used to compute `LHSum`, `MPSum`, and `RHSum` for $n = 4$ on these graphs.)

**4**. Continue by typing the command line below into Maple. This command instructs Maple to compute the actual area $\int_{x=a}^{b} f(x)\,dx$.

```
> Int(f(x),x=a..b) = evalf(int(f(x),x=a..b));
```

At this time make a hard-copy of your typed input and Maple's responses. Then:

**5**. On the graphs you produced in Task 3, draw the rectangles, and plot and label the sample points $(x_i^*, f(x_i^*))$, used to compute `LHSum`, `RHSum`, and `MPSum` for $n = 4$.

***Comments***

1. Over the interval $[1, 3]$, $f(x) = x^2 + x$ is increasing (can you see how you might verify this without graphing?); thus the right-hand Riemann sum is associated with *circumscribed* rectangles, and the left-hand Riemann sum is associated with *inscribed* rectangles. Over the interval $[-3, -1]$, on the other hand, $f(x) = x^2 + x$ is decreasing (can you see how you might verify this without graphing?); thus here the right-hand Riemann sum is associated with *inscribed* rectangles, and the left-hand Riemann sum is associated with *circumscribed* rectangles.
2. Observe that the values of `LHSum`, `RHSum` and `MPSum` are not necessarily the same for any finite $n$, but they get closer and closer to each other as $n$ gets larger and larger, and their limits (as $n$ goes to $\infty$) are *all* the same.