

Inverse Functions

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Objective

To investigate some properties of inverse functions.

Narrative

In this project we discuss inverse functions. Some of the key things you should know about inverse functions are:

1. To show that a function f has an inverse, you need to show that f is 1-1. You can do this by showing that either: a) $f'(x) > 0$ for all x in the domain of f , or b) $f'(x) < 0$ for all x in the domain of f .
2. You can find the inverse f^{-1} of a simple function f by solving the equation $y = f(x)$ for x in terms of y ; the resulting equation is $x = f^{-1}(y)$. (So if you are looking for $f^{-1}(x)$, simply interchange x and y in this equation.) Remember that this only works for simple functions f .
3. One way to check your work in computing the inverse f^{-1} of a simple function f is to verify that: a) $f^{-1}(f(x)) = x$ for all x in the domain of f , and b) $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} .
4. The graph of the inverse f^{-1} of a function f is the reflection of the graph of f in the line $y = x$.
5. The derivative of f is related to the derivative of f^{-1} by the equation $D_x(f^{-1}(x)) = 1/D_y(f(y))$.

In this project we investigate the function $f(x) = x/(3-x)$ and its inverse. In this project we also use the command `unapply`: If y is an expression in x then `unapply(y,x)` is the function f for which $y = f(x)$. For example, if $y = 3x^2 + 4$ then `unapply(y,x)` is the function $f() = 3()^2 + 4$.

Task

1. a) Type the command lines in the left-hand column below into Maple in the order in which they are listed. They begin by defining $f(x) = x/(3-x)$ and checking that it does have an inverse.

```
> # Your name, today's date
> # Inverse Functions
> restart;                               Clear Maple's memory.
> # Task 1a
> f := x -> x/(3-x);                     Let  $f(x) = x/(3-x)$ .
> simplify(D(f)(x));                     Since  $f'(x) > 0$  for all  $x$ ,  $f$  has an inverse.
```

- b) Continue by typing the following command lines. They find f^{-1} and check our work.

```
> # Task 1b
> sol := solve(y=f(x),x);                 Let sol denote the expression we get from solving
                                          the equation  $y = f(x)$  for  $x$  in terms of  $y$ .
> g := unapply(sol,y);                   Use the expression sol to define the function  $g = f^{-1}$ .
> simplify(f(g(x)));                     Check that  $(f \circ g)(x) = x$ .
> simplify(g(f(x)));                     Check that  $(g \circ f)(x) = x$ .
```

- c) Continue by typing the following command lines. They graph f , f^{-1} , and the line $y = x$.

```
> # Task 1c
> plot({f(x),g(x),x},x=0..2,y=0..2,scaling=constrained);
```

- d) Continue by typing the following command lines. They compute the derivative $D_x(g(x))$ of $g = f^{-1}$ with respect to x , and then check the computation by verifying that $D_x(g(x)) = 1/D_y(f(y))$. (Observe that

since the left-hand side of this equation is in x and the right-hand side is in y we must use the substitution $y = g(x)$ to relate x and y . Why don't we use $y = f(x)$ instead? The reason is that we interchanged variables when we started writing g as a function of x rather than of y ; thus we could have used the substitution $x = f(y)$, but this is the same as the equation $y = g(x)$!

```
> # Task 1d
> simplify(D(g)(x));
> simplify(1/D(f)(y));
> subs(y=g(x),%);
> simplify(%);
```

At this point, make a hard-copy of your typed input and Maple's responses (both text and graphics). Then:

2. Label by hand the graph of the line whose equation is $y = x$ on the plot you created in Task 1, as well as the graphs of f and g .
3. The point $P(1, \frac{1}{2})$ lies on the graph of f , and the point $Q(\frac{1}{2}, 1)$ lies on the graph of f^{-1} . Plot and label by hand P and Q . (These points should be mirror reflections of each other in the graph of the line $y = x$.)

Your lab report will be a hard-copy of your typed input and Maple's responses (both text and hand-drawn graphics).

Comments

You might find it instructive to repeat this project for the functions $f(x) = 2x + 1$, $f(x) = \ln x$, and $f(x) = e^x$. In Maple, the natural log function is written `ln(x)`, the natural exponential function is written `exp(x)`, and the common log function is written `log10(x)`. You might also find it instructive to repeat this project for the function $f(x) = x^3$; this example is a little different than the others we have considered so far, however: in this case Maple returns *three* expressions when it solves the equation $y = f(x)$ for x in terms of y . Can you see why? In this case you need to use `g := unapply(sol[1],y)` rather than `g := unapply(sol,y)` to make the rest of the code work.