The Exponential Function and Interest

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Objective

To discuss the fact that $e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$ in the context of an application to finance.

Narrative

If a principal of P_0 is invested at an annual percentage rate (APR) of r (expressed as a decimal) compounded n times a year then after T years the investment is worth

$$P_0 \left(1 + \frac{r}{n}\right)^{nT} \tag{1}$$

For example, if \$1000 is invested at an APR of 6% compounded 12 times a year then after T years the investment is worth

$$\$1000 \left(1 + \frac{0.06}{12}\right)^{12T} = \$1000(1.005)^{12T}.$$

(You might find it useful to think of this quantity as \$1000 times $12T$ factors of 1.005: one factor for each month the principal is invested.)

More precisely, at *any* time t the value of the investment is determined by (1) *at the end of* the previous compounding period. For example, if a principal of P_0 is invested at a APR of r% compounded 12 times a year then:

1. Prior to the the end of the first month, the investment is (still) worth P_0.
2. At any time during the second month, the investment is worth $P_0 \left(1 + \frac{r}{12}\right)^1$.
3. At any time during the third month, the investment is worth $P_0 \left(1 + \frac{r}{12}\right)^2$.

It follows that the value of the investment is actually determined by a piecewise-defined function in t. In this project we illustrate this, and the connection between this and the exponential function, graphically.

Tasks

1. Type the command lines below into Maple in the order in which they are listed; they create graphics of $f(x) = e^x$ and five hypothetical interest functions $f_n(x)$ (corresponding to compounding $n = 1, 2, 4, 8$, and 16 times per year, respectively) over 3 years, as described above.

```
> # Your name, today´s date
> # The Exponential Function and Interest
> restart;
> with(plots):
> plot0 := plot(exp(x),x=0..3):
> display(plot0);
> plot1 := plot(exp(trunc(x)),x=0..3,color=black,thickness=3,discont=true):
> display({plot0,plot1});
> plot2 := plot(exp(trunc(2*x)/2),x=0..3,color=black,thickness=3,discont=true):
> display({plot0,plot2});
> plot3 := plot(exp(trunc(4*x)/4),x=0..3,color=black,thickness=3,discont=true):
> display({plot0,plot3});
> plot4 := plot(exp(trunc(8*x)/8),x=0..3,color=black,thickness=3,discont=true):
> display({plot0,plot4});
> plot5 := plot(exp(trunc(16*x)/16),x=0..3,color=black,thickness=3,discont=true):
> display({plot0,plot5});
```

Observe how the interest functions $f_n(x)$ approach the exponential function as the number of compounding periods increases.

At this time, make a hard-copy of your typed input and Maple's responses. Then:

2. In each of the graphics you created in Task 1, label by hand the graphs of $f(x) = e^x$ and $f_n(x)$; in the last graphic, for example, label the graph of $f(x) = e^x$ by "$f(x) = e^x$", and the graph of $f_{16}(x)$ by "$f_{16}(x)$".

3. How much is \$1000 worth if it is invested at an APR of 9%:

a) compounded quarterly (every 3 months), at the end of 3 months, 6 months, 8 months, 1 year, 16 months, 2 years?

b) compounded continuously at the end of 3 months, 6 months, 8 months, 1 year, 16 months, 2 years?

Your lab report will be a hard copy of your typed input and Maple's responses (both text and hand-labeled graphics).

Comments

The point of this project is that when we discuss compound interest as a function of time t, t can either be interpreted as a discrete variable whose values correspond to the times at which interest is compounded, or as a continuous variable. And, in the later case, if you invested money in an account that earns interest compounded n times a year, then the value of that account *at any time* during the year is determined by a *piecewise-defined* function $f_n(t)$ defined for all t, and the sequence of functions $f_n(t)$ approaches an exponential function as $n \to \infty$.