

# Exponentials and Logarithms

Michael Penna, Indiana University – Purdue University, Indianapolis

## Objective

To briefly review some basic algebraic properties of exponentials and logarithms. This project involves computations you can make by hand.

## Exponentials

Recall that for any real number  $a$ , the exponential  $a^p$  is defined as follows:

$$a^p = \begin{cases} \underbrace{a \cdot a \cdots a}_{n \text{ times}} & \text{if } p = n \text{ is a positive integer} \\ 1 & \text{if } a \neq 0 \text{ and } p = 0 \\ \sqrt[n]{a} & \text{if } p = 1/n \text{ (and } a > 0 \text{ when } n \text{ is even)} \\ \left(a^{\frac{1}{n}}\right)^m & \text{if } p = m/n \text{ (and } a > 0 \text{ when } n \text{ is even)} \\ 1/a^n & \text{if } a \neq 0 \text{ and } p \text{ is negative} \end{cases}$$

In particular, recall that:

1.  $a^p$  is defined only for quotients of integers, or rational numbers,  $p$  (so, for example,  $3^{\sqrt{2}}$  is not defined since  $\sqrt{2}$  cannot be expressed as a quotient of integers),
2.  $a^0 = 1$  if  $a \neq 0$ , and  $0^0$  is not defined, and
3.  $ca^p$  means  $c \cdot a^p$ , not  $(ca)^p$ .

Exponentials satisfy the following properties:

1.  $a^p a^q = a^{p+q}$  (so, for example,  $2^3 2^4 = 2^7$ ),
2.  $\frac{a^p}{a^q} = a^{p-q}$  (so, for example,  $\frac{2^7}{2^4} = 2^3$ ),
3.  $(a^p)^q = a^{pq}$  (so, for example,  $(2^3)^4 = 2^{12}$ ),
4.  $(ab)^p = a^p b^p$  (so, for example,  $(2 \cdot 3)^4 = 2^4 3^4$ ), and
5.  $\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$  (so, for example,  $\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4}$ ).

## Logarithms

If  $B$  is any real number (any *base*) other than 1, we define *the logarithm*  $\log_B a$  of  $a$  to the base  $B$  to be the *rational number*  $p$ , if it exists, for which  $B^p = a$ . In other words,

$$p = \log_B a \quad \text{if and only if} \quad B^p = a.$$

Note that the logarithm of a number is always a *rational* number (since exponentials were only defined for the case in which the exponent is rational).

For example,  $\log_3 9 = 2$  since  $3^2 = 9$ . And if you want to find  $\log_5 625$ , ask yourself the question, “For what value of  $p$  is  $5^p = 625$ ?” In other words, remember that

$$p = \log_5 625 \quad \text{if and only if} \quad 5^p = 625.$$

Once you observe that  $5^4 = 625$ , you will know that  $\log_5 625 = 4$ . One big difference between exponentials and logarithms is that computing exponentials is a direct process, while computing logarithms is an indirect — or inverse — process: to compute (or answer any sort of question about) logarithms, you often need to translate your question into one of exponentials.

Logarithms satisfy the following properties:

1.  $\log_B B = 1$ ,  $\log_B 1 = 0$ , and  $\log_B 0$  does not exist,
2.  $\log_B(ab) = \log_B a + \log_B b$  (so, for example,  $\log_B 6 = \log_B 2 + \log_B 3$ ),
3.  $\log_B \frac{a}{b} = \log_B a - \log_B b$  (so, for example,  $\log_B \frac{2}{3} = \log_B 2 - \log_B 3$ ), and
4.  $\log_B(a^b) = b \log_B a$  (so, for example,  $\log_B(2^3) = 3 \log_B 2$ ).

On the other hand, note that:

1.  $\log_B(ab) \neq (\log_B a)(\log_B b)$ ,
2.  $\log_B \frac{a}{b} \neq \frac{\log_B a}{\log_B b}$ ,
3.  $\log_B(a^b) \neq (\log_B a)^b$ ,
4.  $\log_B(a + b) \neq \log_B a + \log_B b$ , and
5.  $\log_B(a - b) \neq \log_B a - \log_B b$ .

Finally, note that not all real numbers have logarithms: for example,  $3^{\sqrt{2}}$  does not have a base 3 logarithm  $p$  since there is no *rational* number  $p$  for which

$$p = \log_3 3^{\sqrt{2}} \quad \text{or} \quad 3^p = 3^{\sqrt{2}}.$$

It would be natural to suggest letting  $p = \sqrt{2}$ , but  $\sqrt{2}$  is not a rational number. This may seem like quibbling over a small matter, but it brings us back to the fact that  $a^p$  is not defined if  $p$  is not a rational number.

### Exercises

What is:

1.  $\log_{10} 100$
2.  $\log_2 \frac{1}{32}$
3.  $\log_9 3$
4.  $\log_5 1$
5.  $\log_7 7$
6.  $\log_2 -1$
7.  $3^{\log_3 5}$
8.  $\log_2 2^7$

Solve the following equations for  $x$ :

9.  $2^{-x} = 8$
10.  $\log_6(\log_6 x) = 2$
11.  $10^{2 \log_{10} x} = \frac{1}{25}$
12.  $\log_{10} 10^{5x} = 15$
13.  $\log_3(x + 1) = 2$
14.  $\log_3 x = \log_3(5 - x)$
15.  $\log_{10} x^2 = \log_{10}(-3x - 2)$
16.  $\log_5(2x + 3) = \log_5 11 + \log_5 3$
17.  $\log_2 x + \log_2(x + 2) = 3$
18.  $\log_6(2x - 3) = \log_6 12 - \log_6 3$
19.  $\log_5(x + 2) - \log_5 x = 2 \log_5 4$
20.  $\log_{10} x = 1 - \log_{10}(x + 2)$
21.  $\log_2 x + \log_2(x + 6) = \frac{1}{2} \log_2 9$
22.  $\log_6(x + 5) + \log_6 x = 2$