Linear Approximation

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Objective

In this project we discuss one reason the problem of finding the equation of the tangent line to the graph of a function at a point is important.

Narrative

An important practical problem faced in the programming of computational devices (such as hand-calculators and computers), is how to find the value of functions such as $\sqrt{x}$, $\sin x$, e^x, and $\ln x$. Computational devices are so ubiquitous that you might not think about this, but *someone* has to program these devices, and evaluating functions such as these *is* an issue for anyone involved in writing low-level computer code.

Knowing the equation $y = mx + b$ of the tangent line to the graph of a function f at a given point $x = a$ is important since the associated function $L(x) = mx + b$ is the best linear approximation to f at a, and linear functions are easy to evaluate. (See Comment 1 at the end of this project.) For example, we will later see that if angles are measured in degrees then the tangent line to the graph of $f(x) = \sin x$ at $x = 0$ is the graph of $L(x) = 0.0175x$. Thus for "small" angles x, $\sin x \approx 0.0175x$. So if we're interested in computing sin 3°, for instance,

$$\sin 3^\circ \approx 0.0175 * 3 = 0.0525.$$

Tasks

1. Type the command lines below into Mathematica in the order in which they are listed. They produce the graphs of $y = \sin x$ and $y = x$ at two different scales. Observe how close the graphs of $y = \sin x$ and $y = x$ are to one another!

```
In[1]:= (* Your name, today's date *)
In[2]:= (* Linear Approximation *)
In[3]:= (* Task 1 *)
In[4]:= Plot[{x, Sin[x]}, {x,-Pi/2,Pi/2}, AxesOrigin->{0,0}]
In[5]:= Plot[{x, Sin[x]}, {x,-Pi/6,Pi/6}, AxesOrigin->{0,0}]
```

2. Continue by typing the following command lines into Mathematica. They produce numerical information about the values of $f(x) = \sin x$ and $f(x) = x$ for small values of x (in degrees and in radians) and store them in a matrix (a rectangular array of numbers) **A**.

```
In[6]:= (* Task 2 *)
In[7]:= OneDeg = N[Pi/180]
In[8]:= A = Array[a, {31,4}];
In[9]:= {a[1,1] = "x (degs)", a[1,2] = "x(radians)", a[1,3] = "sin(x)", a[1,4] = "difference"};
In[10]:= For[n=0, n<31, n++,
            {a[n+1,1] = n, a[n+1,2]= n*OneDeg,
              a[n+1,3] = Sin[n*OneDeg], a[n+1,4] = Abs[a[n+1,2]-a[n+1,3]]}];
In[11]:= TableForm[A]
```

Again, observe how close the values for $\sin x$ and x (in radians) are to one another.

At this time make a hard-copy of your typed input and Mathematica's responses. Then:

3. Using the table you created in Task 2, find an approximation to $\sin 5^\circ$ to 4 decimal places of accuracy. Roughly how much error is there in this approximation?

4. Later we will see that the tangent line to the graph of $y = \sqrt{x}$ at $x = 1$ has the equation $y = \frac{1}{2}x + \frac{1}{2}$. Assuming this for now, use the equation of the tangent line to $y = \sqrt{x}$ to approximate $\sqrt{1.1}$ to 4 decimal places of accuracy. Based on the fact that (according to Mathematica) $\sqrt{1.1} = 1.048808848$, roughly how much error is there in your approximation?

Your lab report will be a hard copy of your typed input and Mathematica's responses, and your written responses.

Comments

1. The reason linear functions are easy to evaluate is that they can be evaluated using only the 4 basic arithmetic operations of addition, subtraction, multiplication, and division, and — whatever their other capabilities may be — these operations can be performed by almost all computational devices.
2. Evaluating functions such as $\sqrt{x}$, $\sin x$, e^x, and $\ln x$ is *only one* reason finding the equation of the tangent line to the graph of a function at a point is important. There are *many* others which we will discuss later.
3. Later we will also discuss how small "small" is, and how to handle "large" angles when evaluating functions such as sin.