The Definition of Limit

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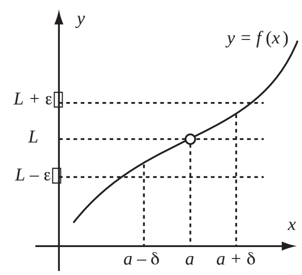
Objective

To investigate the precise definition of limit.

Narrative

To prove that the number L which we *guess* to be the limit of $f(x)$ at $x = a$, *really is* the limit of $f(x)$ at $x = a$, we must verify the condition in the formal definition of limit. This condition requires that for each real number $\epsilon > 0$, there is a real number $\delta > 0$ such that the values of $f(x)$ for all x in the interval $(a - \delta, a + \delta)$ — except possibly at $x = a$ itself — lie between $L - \epsilon$ and $L + \epsilon$; that is,

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon.$$



In this project we investigate this condition.

Task

1. Type the command lines below into Mathematica in the order in which they are listed. These commands are concerned with $\lim_{x\to 2}(-x^3/12 + x^2/2 + 5/3)$.

```
In[1]:= (* Your name, today's date *)
In[2]:= (* The Definition of Limit *)
In[3]:= f[x_] := -x^3/12+x^2/2+5/3
In[4]:= a = 2
In[5]:= L = Limit[f[x], x->a]
In[6]:= Epsilon = 0.5
In[7]:= Plot[{f[x], L, L+Epsilon, L-Epsilon}, {x,0,3}, AxesOrigin->{0,0}]
```

2. Continue by typing the command lines below into Mathematica in the order in which they are listed.

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In[8]:= Epsilon = 0.2
In[9]:= Plot[{f[x], L, L+Epsilon, L-Epsilon}, {x,0,3}, AxesOrigin->{0,0}]
```

At this time make a hard-copy of your typed input and Mathematica's responses. Then:

3. Label by hand the graphs of $y = f(x)$, $y = L$, $y = L \pm \epsilon$, and $x = a$ on the graphic you produced in Task 1. Estimate by eye and state a value of δ for which the values of $f(x)$ for all $x \in [a - \delta, a + \delta]$ — except possibly at a — lie between $L - \epsilon$ and $L + \epsilon$ when $\epsilon = 0.5$. Draw the lines whose equations are $x = a + \delta$ and $x = a - \delta$ by hand on the graphic you drew in Task 1, and highlight that part of the graph of f for which $x \in (a - \delta, a + \delta)$.

4. Repeat Task 3 for the graphic you produced in Task 2, now letting $\epsilon = 0.2$.

Your lab report will be a hard copy of your typed input and Mathematica's responses (both text and hand-labeled graphics).

Comments

In this project we are *not* actually proving that $L = \lim_{x\to a} f(x)$. On one hand, we are just verifying that an appropriate δ exists for *two* given ϵ's: to verify that $L = \lim_{x\to a} f(x)$, we would have to do this *for every*

ϵ, not just two, three, four, or any finite number of ϵ's. On the other hand, since Mathematica draws the graphs of functions by "connecting-the-dots", some significant behavior could occur *between* the dots that is not revealed by Mathematica, so we cannot trust Mathematica's graphics to be completely accurate, either. This is one of the big reasons the theoretical $\epsilon\delta$-analysis of limits is so important.