The Bisection Method

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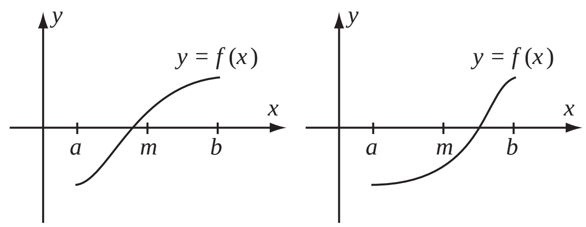
Objective

In this project we discuss the Bisection Method. The Bisection Method is a simple method for solving equations that is a consequence of the Intermediate Value Theorem.

Narrative

The Bisection Method is a technique for finding an approximation to a solution of the equation $f(x) = 0$, where f is a continuous real-valued function, given values a and b of x for which $f(a)$ and $f(b)$ have opposite signs. The Bisection Method is an iterative algorithm. It is based on the fact that if $m = (a + b)/2$ is the midpoint of $[a, b]$ and $f(a)$ and $f(m)$ have opposite signs then a solution to $f(x) = 0$ lies between a and m, and if $f(b)$ and $f(m)$ have opposite signs (or $f(a)$ and $f(m)$ have the same sign) then a solution to $f(x) = 0$ lies between b and m.



Task

1. Type the command lines below into Mathematica in the order in which they are listed. They apply the Bisection Method to determining where $f(x) = x^2 - 2 = 0$ assuming $a = 1$ and $b = 2$. (The resulting zero of f is $\sqrt{2}$.) In addition to using a Mathematica **For** loop, this code uses a Mathematica **If** statement. An **If** statement in Mathematica has the general form:

If[*condition, true consequence, false consequence*]

Note that since $f(x) = x^2 - 2$ is continuous, $f(1) < 0$ and $f(2) > 0$, so the Intermediate Value Theorem assures us that there will be a value of x for which $f(x) = 0$.

```
In[1]:=   (* Your name, today´s date *)
In[2]:=   (* The Bisection Method *)
In[3]:=   (* Task 1 *)
In[4]:=   f[x_] = x^2-2
In[5]:=   {a=1, b=2}
In[6]:=   For[n=1, n<10, n++,
               {m=(a+b)/2,
                If[f[a]*f[m]<0, b=m, a=m],
                Print[N[a],"      ", N[m], "      ", N[b]]
               }
             ]
```

(The reason for writing `In[6]` as we did above, is to make it easier to read. Writing code this way is a standard practice in computer programming when the code becomes difficult to read. We did it here for the purpose of illustration.)

The logic behind the **If** statement in the above code is that if $f(a)f(m) < 0$ then $f(a)$ and $f(m)$ have the different signs, in which case by redefining b to be m, we are ready to iterate our procedure; if $f(a)f(m) > 0$

then $f(a)$ and $f(m)$ have the same sign, in which case by redefining b to be m, we are ready to iterate our procedure.

From the data created by the above code we see that the solution of $x^2 - 2 = 0$ is eventually between 1.41420 and 1.41422. Thus, to 4 decimal places of accuracy, the solution is 1.4142.

3. The equation $x^3 + x - 1 = 0$ has just one real solution. Use the Bisection Method to approximate this solution to 4 decimal places of accuracy. (Caution: If you use command lines similar to those you used in Task 1 then you will have to make different choices for a and b! To determine appropriate values of a and b, you might want to look at a graph of f.)

At this time make a hard-copy of your typed input and Mathematica's responses: it will be your lab report.

Comments

1. There are numerous ways in which we might want to improve our implementation of the Bisection Method if we were to be making continued use of it. We will not make continued use of it, however, since we will soon introduce a superior technique — Newton's Method — which is based on the concept of differentiation.
2. In applying the Bisection Method we produce a sequence of numbers which has the property that more and more digits "stabilize" as we go further and further in the sequence. This is the essence of the limiting process. If you learned the Bisection Method prior to taking Calculus, did you know that you were actually studying a limiting process?!
3. Later we will be able to easily prove that the equation $x^3 + x - 1 = 0$ has just one real solution.