

Finding Relative Extrema on an Open Interval: The First Derivative Test

Michael Penna, Indiana University – Purdue University, Indianapolis

Objective

To illustrate how to find and classify the relative extrema of a function on an open interval using the First Derivative Test.

Narrative

Recall that to find the maximum and minimum values of any function $f(x)$ on an open interval using the First Derivative Test we:

1. find the critical numbers of f (the values of x for which $f'(x) = 0$ and $f'(x)$ does not exist),
2. compute the sign of the first derivative f' of f at points x_0 and x_1 just to the left and right of each critical number c ($x_0 < c < x_1$): if
 - (a) $f'(x_0) > 0$ and $f'(x_1) < 0$ then $f(c)$ is a relative maximum,
 - (b) $f'(x_0) < 0$ and $f'(x_1) > 0$ then $f(c)$ is a relative minimum,
 - (c) $f'(x_0) > 0$ and $f'(x_1) > 0$, or $f'(x_0) < 0$ and $f'(x_1) < 0$, then $f(c)$ is neither a relative maximum nor a relative minimum. (In this last case, the point $P(c, f(c))$ is a saddle point.)

In this project we illustrate this process as we find the relative maximum and minimum values of $f(x) = x(1 - x^2)^{2/3}$.

Tasks

1. a) Type the command lines in the left-hand column below into Mathematica in the order in which they are listed. The effect of each command is described in the right-hand column for your reference.

In[1] := (* Your name, today's date *)	
In[2] := (* The First Derivative Test *)	
In[3] := f[x_] := x((1-x^2)^2)^(1/3)	Let $f(x) = x(1 - x^2)^{2/3}$. (See below.)
In[4] := f'[x]	Here's $f'(x)$. Observe that it's a quotient.
In[5] := f1=Simplify[%]	Simplify $f'(x)$.
In[6] := Solve[Numerator[f1]==0,x]	Find where the numerator of $f'(x) = 0$.
In[7] := Solve[Denominator[f1]==0,x]	Find where the denominator of $f'(x) = 0$.

The reason we wrote $x(1 - x^2)^{2/3}$ as $x((1-x^2)^2)^(1/3)$ in the above code is that when $|x| > 1$, $1 - x^2 < 0$ and in this case Mathematica computes $(1 - x^2)^{2/3}$ by taking the cube root of $1 - x^2$ — which it views as a complex number — and squaring — yielding another complex number (which is something we don't want: in this course we're restricting our attention to the real numbers). In writing $x((1-x^2)^2)^(1/3)$ we are forcing Mathematica to square $1 - x^2$ first — yielding a real number — and then take the cube root — yielding another real number.

- b) Use Mathematica to evaluate f' at points x_0 and x_1 just to the left and right of each critical number.
- c) On the basis of your computations, write Mathematica comments stating whether $f(c)$ is a relative maximum or relative minimum, or whether $P(c, f(c))$ is a saddle point, for each critical number c . (Your comment might look like, “(* is a relative maximum*)” for example.)
- d) Type the following command line into Mathematica, substituting for **a** and **b** numbers that are large enough to enclose all the critical numbers of f and the graph of f over $[a, b]$.

```
In[8] := Plot[f[x],{x,a,b}]
```

At this time make a hard-copy of your typed input and Mathematica's responses. Then:

2. By hand, on the graphic you produced in Task 1, label the graph of f , plot and label the critical numbers of f along the x -axis, and plot and label the points $P(c, f(c))$, c a critical number of f . Remembering that the relative extrema of f are y -values, estimate the relative extrema of f by eye and for each relative extremum write a sentence under your graphic of the form, " f has a relative maximum of $_$ when $x = _$ " or " f has a relative minimum of $_$ when $x = _$ ".

Your lab report will be a hard-copy of your typed input and Mathematica's responses (both text and hand-labeled graphics).