Finding Relative Extrema on an Open Interval: The Second Derivative Test

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Objective

To illustrate how to find and classify the relative extrema of a function on an open interval using the Second Derivative Test.

Narrative

Recall that to find the maximum and minimum values of any function $f(x)$ on an open interval using the Second Derivative Test we:

1. find the critical numbers of f (the values of x for which $f'(x) = 0$ and $f'(x)$ does not exist),
2. compute the sign of the second derivative f'' of f at each critical number c for which $f'(c) = 0$: if
   (a) $f''(c) > 0$ then $f(c)$ is a relative minimum,
   (b) $f''(c) < 0$ then $f(c)$ is a relative maximum,
   (c) $f''(c) = 0$ then the test fails, and we must resort to the First Derivative Test: we compute the sign of the first derivative f' of f at points x_0 and x_1 just to the left and right of the critical number c ($x_0 < c < x_1$); if
      i. $f'(x_0) > 0$ and $f'(x_1) < 0$ then $f(c)$ is a relative maximum,
      ii. $f'(x_0) < 0$ and $f'(x_1) > 0$ then $f(c)$ is a relative minimum,
      iii. $f'(x_0) > 0$ and $f'(x_1) > 0$, or $f'(x_0) < 0$ and $f'(x_1) < 0$, then $f(c)$ is a neither a relative maximum nor a relative minimum. (The point $P(c, f(c))$ is a saddle point.)

In this project we illustrate this process as we find the relative maximum and minimum values of $f(x) = x(1 - x^2)^{2/3}$.

Tasks

1. a) Type the command lines in the left-hand column below into Mathematica in the order in which they are listed. The effect of each command is described in the right-hand column for your reference.

`In[1]:=` **(* Your name, today's date *)**	
`In[2]:=` **(* The Second Derivative Test *)**	
`In[3]:=` **f[x_] := x((1-x^2)^2)^(1/3)**	Let $f(x) = x(1 - x^2)^{2/3}$. (See below.)
`In[4]:=` **f´[x]**	Here's $f'(x)$. Observe that it's a quotient.
`In[5]:=` **f1 = Simplify[%]**	Simplify $f'(x)$.
`In[6]:=` **Solve[Numerator[f1]==0,x]**	Find where the numerator of $f'(x) = 0$.
`In[7]:=` **Solve[Denominator[f1]==0,x]**	Find where the denominator of $f'(x) = 0$.
`In[8]:=` **f´´[x]**	Here's $f''(x)$.

The reason we wrote $x(1-x^2)^{2/3}$ as **x((1-x^2)^2)^(1/3))** in the above code is that when $|x| > 1$, $1-x^2 < 0$ and in this case Mathematica computes $(1 - x^2)^{2/3}$ by taking the cube root of $1 - x^2$ — which it views as a complex number — and squaring — yielding another complex number (which is something we don't want: in this course we're restricting our attention to the real numbers). In writing **x((1-x^2)^2)^(1/3))** we are forcing Mathematica to square $1 - x^2$ first — yielding a real number — and then take the cube root — yielding another real number.

b) Use Mathematica to evaluate f'' at each critical number c for which $f'(c) = 0$. On the basis of your computations, write Mathematica comments stating whether $f(c)$ is a relative maximum or relative minimum, or whether $P(c, f(c))$ is a saddle point, for each critical number c. (Your comment might look like, "**(* _ is a relative maximum *)**" for example.) If $f'(c)$ does not exist or $f''(c) = 0$, use the First Derivative Test at c to determine whether $f(c)$ is a relative maximum or a relative minimum, or whether $P(c, f(c))$ is a saddle point.

c) Type the following command line into Mathematica, substituting for `a` and `b` numbers that are large enough to enclose all the critical numbers of f and the graph of f over $[a, b]$.

```
In[9]:= Plot[f[x], {x,a,b}]
```

At this time make a hard-copy of your typed input and Mathematica's responses. Then:

2. By hand, on the graphic you produced in Task 1, label the graph of f, plot the critical numbers of f along the x-axis, and plot and label the points $P(c, f(c))$, c a critical number of f. Remembering that the relative extrema of f are y-values, estimate the relative extrema of f by eye and for each relative extremum write a sentence under your graphic of the form, "f has a relative maximum of _ when $x =$ _" or "f has a relative minimum of _ when $x =$ _".

Your lab report will be a hard-copy of your typed input and Mathematica's responses (both text and hand-labeled graphics).