Concavity and Inflection Points

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Objective

To illustrate how to analyze the concavity and find the inflection points of a function.

Narrative

The graph of a function f is concave up at x if $f''(x) > 0$ and concave down at x if $f''(x) < 0$, and an *inflection point* of f is a point $P(x, f(x))$ on the graph of f at which the concavity of f changes. To find the inflection points of f, we first find the values of x for which $f''(x) = 0$; these are the *potential inflection numbers* of f. We then look at the value of the second derivative just to the left and just to the right of each potential inflection number x: if the values have different signs then x is *an inflection number*, and $P(x, f(x))$ is an *inflection point* of f; if the values have the same sign then x is not an inflection number, and $P(x, f(x))$ is not an inflection point of f.

In this project we analyze the concavity and find the inflection points of $f(x) = x(1 - x^2)^{2/3}$.

Task

1. Type the command lines in the left-hand column below into Maple in the order in which they are listed. The effect of each command is described in the right-hand column for your reference.

`In[1]:=` **(* Your name, today's date *)**	
`In[2]:=` **(* Concavity and Inflection Points *)**	
`In[3]:=` **f[x_] := x((1-x^2)^2)^(1/3)**	Let $f(x) = x(1 - x^2)^{2/3}$.
`In[4]:=` **f´[x]**	Here's $f'(x)$. Observe that it's a quotient.
`In[5]:=` **f´´[x]**	Here's $f''(x)$. Observe that it's also a quotient.
`In[6]:=` **f2=Simplify[%]**	Simplify $f''(x)$.
`In[7]:=` **Solve[Numerator[f2]==0,x]**	Find where the numerator of $f''(x) = 0$.
`In[8]:=` **Solve[Denominator[f2]==0,x]**	Find where the denominator of $f''(x) = 0$.

The reason we wrote $x(1-x^2)^{2/3}$ as **x((1-x^2)^2)^(1/3))** in the above code is that when $|x| > 1$, $1-x^2 < 0$ and in this case Mathematica computes $(1 - x^2)^{2/3}$ by taking the cube root of $1 - x^2$ — which it views as a complex number — and squaring — yielding another complex number (which is something we don't want: in this course we're restricting our attention to the real numbers). In writing **x((1-x^2)^2)^(1/3))** we are forcing Mathematica to square $1 - x^2$ first — yielding a real number — and then take the cube root — yielding another real number.

b) Use Maple to evaluate f'' at points x_0 and x_1 just to the left and right of each potential inflection number of f.

c) On the basis of your computations, write Maple comments stating whether $P(c, f(c))$ is an inflection point, for each potential inflection number c. (Your comment might look like, "**(* ____ is an inflection point *)**" for example.)

d) Type the following command line into Mathematica, substituting for `a` and `b` numbers that are large enough to enclose all the potential inflection numbers of f over $[a, b]$.

`In[10]:=` **Plot[f[x],{x,a,b}]**
`In[11]:=` **Plot[{f[x], f´´[x]}, {x,a,b}, PlotStyle->{RGBColor[1,0,0],RGBColor[0,1,0]}]**

At this time make a hard-copy of your typed input and Maple's responses. Then:

2. By hand, on the first graphic you produced in Task 1, label the graph of f, plot the potential inflection numbers of f along the x-axis, and plot and label the points $P(c, f(c))$ for c a potential inflection number of f. For each inflection *point* write a sentence under your graphic of the form, "____ is an inflection point".

3. By hand, on the second graphic you produced in Task 1, label the graphs of f and f''. Then highlight that part of the graph of f which is concave up, and that part of the graph of f'' over which f'' is positive.

Your lab report will be a hard-copy of your typed input and Maple's responses (both text and hand-labeled graphics).