The Iteration Method

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Objective

In addition to the Bisection Method and Newton's Method, we can often obtain an approximate solution to the equation $f(x) = 0$ by using the Iteration Method. The Iteration Method often arises in applications such as economics in the form of "cobweb cycles", and is frequently used because of the simplicity in programming it. In this project we discuss the Iteration Method.

Narrative

To find an approximate solution to the equation $f(x) = 0$ by the Iteration Method, we look for a value x_∞ of x for which $g(x_\infty) = x_\infty$ for an appropriate function $g(x)$; an x for which $g(x) = x$ is called a *fixed point* of g. For example, to find an approximate solution to the equation $x^2 - 2 = 0$ we might choose an arbitrary value for $x_0 \in [1, 1.99]$ and repeatedly compute x_n using the formula

$$x_n = g(x_{n-1}), \quad n = 1, 2, 3, \ldots$$

where $g(x) = -\frac{1}{2}(x^2 - 2) + x$. The reason the Iteration Method works is that

$$g(x) = -\frac{1}{2}(x^2 - 2) + x = x \quad \text{if and only if} \quad x^2 - 2 = 0.$$

The reason we didn't simply let $g(x) = x^2 - 2 + x$ is that with this choice of g, the Iteration Method does not converge. This is where differentiation enters into our study.

The condition that insures the Iteration Method converges is:

$$|g'(x)| < 1 \text{ for all } x \text{ in some open interval } I \text{ containing the fixed point } x_\infty \text{ of } g.$$

The reason this condition insures that the Iteration Method converges is that if $g(x_\infty) = x_\infty$ and $x \in I$ then

$$|g(x) - x_\infty| = |g(x) - g(x_\infty)| = \left|\frac{g(x) - g(x_\infty)}{x - x_\infty}\right| |x - x_\infty| \approx |g'(x_\infty)|\ |x - x_\infty| < |x - x_\infty|$$

so $g(x)$ is closer to x_∞ than x is to x_∞.

One strength of the Iteration Method is that *after* an appropriate function g and an appropriate initial value x_0 are identified, it is fairly easy to implement. One weakness of it is that given f you have to come up with g, and that's not always easy.

Tasks

1. a) Type the commands below into Mathematica in the order in which they are listed. These commands illustrate that the first method described in the Narrative *converges*. (Note how little it takes to implement this method!)

```
In[1]:= (* Your name, today's date *)
In[2]:= (* The Iteration Method *)
In[3]:= (* Task 1a *)
In[4]:= g[x_] := -(x^2-2)/2+x
In[5]:= x = g[1.5]
In[6]:= For[n=0, n<10, n++, {x=g[x], Print[n, "     ", N[x,8]]}]
```

b) Continue by typing the following commands into Mathematica. These command lines illustrate that the second method described in the Narrative *diverges*.

```
In[7]:= (* Task 1b *)
In[8]:= g[x_] := x^2-2+x
In[9]:= x = g[1.5]
In[10]:= For[n=0, n<10, n++, {x=g[x], Print[n, "     ", N[x,8]]}]
```

At this time make a hard-copy of your typed input and Mathematica's responses. Then:

c) Show by hand that if $g(x) = -\frac{1}{2}(x^2 - 2) + x$ then $|g'(x)| = |-x + 1| < 1$ for all $x \in [1, 1.99]$.

d) Show by hand that if $g(x) = (x^2 - 2) + x$ then $|g'(x)| = |2x + 1| \not< 1$ for any $x \in [1, 1.99]$.

2. Use the Iteration Method to find a solution to the equation $\sin(\cos x) = -x$ accurate to 8 decimal places.

3. Use the Iteration Method to find a solution to the equation $x^3 - x - 1 = 0$ accurate to 8 decimal places.

4. The ancient Babylonians had an "averaging iteration" method

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{A}{x_n}\right)$$

for yielding a sequence $x_1, x_2, \ldots$ of successive approximations to $\sqrt{A}$.

a) Show that if this method converges then it converges to $\sqrt{A}$.
b) Implement this method.
c) For what initial values x_1 does this method converge? Justify your answer.

Your lab report will be a hard copy of your typed input and Mathematica's responses (both text and hand-labeled graphics).

Comments

1. As indicated above, one of the strengths of the Iteration Method is that after setting up a problem, it is very easy to program. On the other hand, since setting up a problem includes finding a function g satisfying the convergence condition, and since it is not always "obvious" how to do this, getting through the "set-up" of a problem can be a weakness of the Iteration Method.
2. The Iteration Method can be extended to multiple variables.
3. The Iteration Method is one of the reasons mathematicians are interested in studying what is known as "fixed point theory" .