Integration: Riemann Sums

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Objective

To investigate the approximation of area using Riemann sums.

Narrative

Recall that if $f(x) \geq 0$ for $x \in [a,b]$ then we may approximate the area of the region bounded by the graph of f, $x = a$, $x = b$, and the y-axis by:

1. subdividing $[a,b]$ into n subintervals, each of length $\Delta x = (b-a)/n$,
2. choosing an "sample point" x_i^* in each subinterval, and
3. computing the sum $\Sigma_{i=1}^{n} f(x_i^*)\Delta x$.

In this project we will use Mathematica to illustrate these computations. In this project we use the command:

Sum[E[i], {i,1,n}] sum the expression $E(i)$ from $i = 1$ to $i = n$.

Task

1. Type the command lines in the left-hand column below into Mathematica in the order in which they are listed. These commands are aimed at approximating the area under the graph of $f(x) = x^2 + x$ from $x = 1$ to $x = 3$ using left-hand endpoints. The effect of each command is described in the right-hand column for your reference.

`In[1]:=` **(* Your name, today's date *)**	
`In[2]:=` **(* Integration: Riemann Sums *)**	
`In[3]:=` **(* Task 1 *)**	
`In[4]:=` **f[x_] := x^2+x**	Let $f(x) = x^2 + x$.
`In[5]:=` **{a=1, b=3}**	Let $a = 1$ and $b = 3$.
`In[4]:=` **For[n=1, n<101, n=n+20,**	For each 20th integer n between 1 and 101, ...
{Deltax = (b-a)/n;	let **Deltax** be $(b-a)/n$ ...
LHSum = 0.0;	and **LHSum** be 0.0.
For[i=1, i<n+1, i++,	For each subinterval ...
{x = a +(i-1)*Deltax;	let **x** be the left-hand endpoint, and ...
LHSum = LHSum+f[x]*Deltax;}];	add the area of the ith rectangle to **LHSum**.
Print[n," ", N[LHSum,6]];	Report **LHSum**.
}	
]	

Observe that we use **x** in the above code to denote the x-coordinate of the ith "sample point" $(x_i^*, f(x_i^*)), i = 1, \ldots, n$. And recall that to compute:

- the left-hand sum **LHSum** we use $x_i^* = a + (i-1)\Delta x$,
- the right-hand sum **RHSum** we use $x_i^* = a + i\ \Delta x$, and
- the midpoint-hand sum **MPSum** we use $x_i^* = a + \dfrac{(i-1)+i}{2}\Delta x = a + \dfrac{2i-1}{2}\Delta x$

(x_i^* for **MPSum** being the average of the x_i^*'s for **LHSum** and **RHSum**).

2. a) Enter the appropriate code for approximating the area under the graph of $f(x) = x^2 + x$ from $x = 1$ to $x = 3$ "`for n from 1 to 101 by 20`" using right-hand endpoints.

b) Enter the appropriate code for approximating the area under the graph of $f(x) = x^2 + x$ from $x = 1$ to $x = 3$ "`for n from 1 to 101 by 20`" using midpoints.

3. Continue by typing the command lines below into Mathematica. They draw *three* copies of the graph of f.

```
In[7]:= (* Task 3 *)
In[8]:= Plot1 = Plot[f[x], {x,a,b}, AxesOrigin->{0,0}]
In[9]:= Show[Plot1]
In[10]:= Show[Plot1]
```

(Later, after you have made a hard copy of your typed input and Mathematica's responses, you will be asked to draw the rectangles and plot the sample points $(x_i^*, f(x_i^*))$ used to compute **LHSum**, **RHSum**, and **MPSum** for $n = 4$ on these graphs.)

4. Continue by typing the command line below into Mathematica. This command instructs Mathematica to compute the actual area $\int_{x=a}^{b} f(x)\, dx$.

```
In[11]:= Integrate[f[x], {x,1,3}]
In[12]:= N[%]
```

At this time make a hard-copy of your typed input and Mathematica's responses. Then:

5. On the graphs you produced in Task 3, draw and lightly shade in the rectangles, and plot and label the sample points $(x_i^*, f(x_i^*))$, used to compute **LHSum**, **RHSum**, and **MPSum** for $n = 4$.

Comments

1. Over the interval $[1, 3]$, $f(x) = x^2 + x$ is increasing (can you see how you might verify this without graphing?); thus the right-hand Riemann sum is associated with *circumscribed* rectangles, and the left-hand Riemann sum is associated with *inscribed* rectangles. Over the interval $[-3, -1]$, on the other hand, $f(x) = x^2 + x$ is decreasing (can you see how you might verify this without graphing?); thus here the right-hand Riemann sum is associated with *inscribed* rectangles, and the left-hand Riemann sum is associated with *circumscribed* rectangles.
2. The values of **LHSum**, **RHSum**, and **MPSum** are not necessarily the same for any finite n, but they get closer and closer to each other as n gets larger and larger, and their limits (as n goes to ∞) are *all* the same.

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