The Fundamental Theorem of Calculus

Michael Penna, Indiana University – Purdue University, Indianapolis

Objective

To discuss an alternate proof of the Fundamental Theorem of Calculus.

Narrative

In this project we present an elementary proof of the Fundamental Theorem of Calculus.

Tasks

1. On a separate piece of paper, justify each labeled step in the following proof that for a differentiable function f over the closed interval $[a, b]$, $\int_a^b f'(x)\ dx = f(b) - f(a)$. (Since all your responses to this part must "fit together", it would be advisable to read through this part before you begin and anticipate in early steps what you will need for later steps.)

Step a: $\int_a^b f'(x)\ dx \quad = \lim_{h\to 0} \sum_{i=1}^N f'(w_i)\ \Delta x_i$

Step b: $\approx \sum_{i=1}^N f'(w_i)\ \Delta x_i$

Step c: $\approx \sum_{i=1}^N \dfrac{f(x_i) - f(x_{i-1})}{\Delta x} \Delta x$

Step d: $= \sum_{i=1}^N (f(x_i) - f(x_{i-1}))$

Step e: $= (f(x_1) - f(x_0)) + (f(x_2) - f(x_1)) + \ldots$
$\qquad + (f(x_{N-1}) - f(x_{N-2})) + (f(x_N) - f(x_{N-1}))$

Step f: $= f(x_N) - f(x_0)$

Step g: $= f(b) - f(a)$

(For example, for Step a, you might write, "This is the definition of the definite integral.".)

2. On a separate piece of paper, justify each labeled step in the following proof that if f is continuous over the interval $[a, b]$ and $F(x) = \int_a^x f(t)\ dt$ for $x \in [a, b]$, then $D_x\left(\int_a^x f(t)\ dt\right) = f(x)$:

Step a: $\int_a^x F'(t)\ dt = F(x) - F(a)$

Step b: $= \int_a^x f(t)\ dt - 0 = \int_a^x f(t)\ dt$

Step c: $\int_a^x (F'(t) - f(t))\ dt = 0$

(See below): $F'(t) - f(t) = 0$

Step d: $F'(x) = f(x)$

Step e: $D_x\left(\int_a^x f(t)\ dt\right) = f(x)$

(You do not have to justify the indicated step, but — for the record — it can be justified by the fact that if $\int_a^x g(t)\ dt = 0$ for all $x \in [a, b]$ then $g(x) = 0$ for all $x \in [a, b]$. To see why this is true, suppose that $g(c) \neq 0$ for some $c \in [a, b]$; indeed, for the sake of argument, let us suppose that $g(c) > 0$. Then, since g is continuous, there is an $\epsilon > 0$ such that $g(t) > 0$ for all $t \in [c - \epsilon, c]$. Since

$$\int_a^c g(t)\ dt = \int_a^{c-\epsilon} g(t)\ dt + \int_{c-\epsilon}^c g(t)\ dt,$$

and since $\int_a^x g(t)\ dt = 0$ for all $x \in [a, b]$ this would imply that $\int_{c-\epsilon}^c g(t)\ dt = 0$; but this is impossible since g is positive on $[c - \epsilon, c]$ so $\int_{c-\epsilon}^c g(t)\ dt$ must be strictly greater than 0.)

Comments

The above proof will be useful later in understanding Euler's Method for solving differential equations. Also, it anticipates the concept of "a telescoping sum" which we will also discuss later.