Inverse Functions

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Objective

To investigate some properties of inverse functions.

Narrative

In this project we investigate some properties of inverse functions. Some of the key things you should know about inverse functions are:

1. To show that a function f has an inverse, you need to show that f is $1-1$. If f is differentiable then you can do this by showing that either: a) $f'(x) > 0$ for all x in the domain of f, or b) $f'(x) < 0$ for all x in the domain of f.

2. You can find the inverse f^{-1} of a simple function f by solving the equation $y = f(x)$ for x in terms of y; the resulting equation is $x = f^{-1}(y)$. (So if you are looking for $f^{-1}(x)$, simply interchange x and y in this equation.) Remember that this only works for simple functions f.

3. One way to check your work in computing the inverse f^{-1} of a simple function f is to verify that: a) $f^{-1}(f(x)) = x$ for all x in the domain of f, and b) $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.

4. The graph of the inverse f^{-1} of a function f is the reflection of the graph of f in the line $y = x$.

5. The derivative of f is related to the derivative of f^{-1} by the equation $D_x(f^{-1}(x)) = 1/D_y(f(y))$.

In this project we investigate the function $f(x) = x/(3-x)$ and its inverse.

Task

1. a) Type the command lines in the left-hand column below into Mathematica in the order in which they are listed. They begin by defining a function f and checking that it does have an inverse.

`In[1]:=` **(* Your name, today's date *)**	
`In[2]:=` **(* Inverse Functions *)**	
`In[3]:=` **(* Task 1a *)**	
`In[4]:=` **f[x_] := x/(3-x)**	Let $f(x) = x/(3-x)$.
`In[5]:=` **Simplify[D[f[x],x]]**	Since $f'(x) > 0$ for all x, f has an inverse.

b) Continue by typing the following command lines. They find f^{-1} and check our work.

`In[6]:=` **(* Task 1b *)**	
`In[7]:=` **Solve[y==f[x],x]**	Solve the equation $y = f(x)$ for x in terms of y.
`In[8]:=` **g[y_] := x /. %[[1]]**	Find the inverse function $g = f^{-1}$.
`In[9]:=` **Simplify[f[g[x]]]**	Check that $(f \circ g)(x) = x$.
`In[10]:=` **Simplify[g[f[x]]]**	Check that $(g \circ f)(x) = x$.

c) Continue by typing the following command lines. They graph f, f^{-1}, and the line $y = x$.

`In[11]:=` **(* Task 1c *)**
`In[12]:=` **Plot[{f[x], g[x], x}, {x,0,2}, AspectRatio->1]**

d) Continue by typing the following command lines. They compute the derivative $D_x(g(x))$ of $g = f^{-1}$ with respect to x, and then check the computation by verifying that $D_x(g(x)) = 1/D_y(f(y))$. (Observe that since the left-hand side of this equation is in x and the right-hand side is in y we must use the substitution $y = g(x)$ to relate x and y. Why don't we use $y = f(x)$ instead? The reason is that we interchange variables when we start writing g as a function of x rather than of y; we could have used the substitution $x = f(y)$, but this is the same as the substitution $y = g(x)$!)

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In[13]:= (* Task 1d *)
In[14]:= Simplify[D[g[x],x]]
In[15]:= Simplify[1/D[f[y],y]]
In[16]:= % /. y->g[x]
In[17]:= Simplify[%]
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At this point, make a hard-copy of your typed input and Mathematica's responses (both text and graphics). Then:

2. Label by hand the graph of the line whose equation is $y = x$ on the plot you created in Task 1, as well as the graphs of f and g.

3. The point $P(1, \frac{1}{2})$ lies on the graph of f, and the point $Q(\frac{1}{2}, 1)$ lies on the graph of f^{-1}. Plot and label by hand both P and Q.

Your lab report will be a hard-copy of your typed input and Mathematica's responses (both text and hand-drawn graphics).

Comments

You might find it instructive to repeat this project for the functions $f(x) = 2x + 1$, $f(x) = \ln x$, and $f(x) = e^x$. In Mathematica, the natural log function is written **Log[x]**, and the natural exponential function is written **Exp[x]**. You might also find it instructive to repeat this project for the function $f(x) = x^3$; this example is a little different than the others we have considered so far, however: in this case Mathematica returns *three* expressions when it solves the equation $y = f(x)$ for x in terms of y. Can you see why?