Approximate Integration

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Objective

To illustrate and compare Riemann sums, the Trapezoidal Rule, and Simpson's Rule.

Narrative

In this project we illustrate and compare Riemann sums, the Trapezoidal Rule, and Simpson's Rule by looking at how they approximate $\int_{x=1}^{2} \frac{dx}{x} = \ln 2$.

Task

1. Type the command lines in the left-hand column below into Mathematica in the order in which they are listed. They provide Mathematica some initial information.

`In[1]:=` **(* Your name, today's date *)**	
`In[2]:=` **(* Approximate Integration *)**	
`In[3]:=` **f[x_] := 1/x**	Let $f(x) = 1/x$.
`In[4]:=` **NInts=6**	Let the number of intervals $NInts = 6$.
`In[5]:=` **{a=1, b=2, dx=(b-a)/NInts}**	Let $a = 1, b = 2$, and $dx = (b-a)/NInts$.
`In[6]:=` **Actual=Integrate[f[x], {x,a,b}]**	What is $\int_{x=a}^{b} f(x)\ dx$?
`In[7]:=` **N[%,6]**	

a) Continue by typing the command lines in the left-hand column below into Mathematica in the order in which they are listed. These commands approximate our integral by Riemann sums.

`In[6]:=` **(* Task 1a *)**	
`In[9]:=` **Riemann=0.0**	Initialize `Riemann` to 0.
`In[10]:=` **For[n=0, n<NInts, n++, {Riemann = Riemann+f[a+n*dx]*dx}]**	Compute the Riemann sum.
`In[11]:=` **Riemann**	Report the Riemann sum.
`In[12]:=` **Plot[f[x], {x,a,b}, AxesOrigin->{0,0}]**	Draw the graph of f.
`In[13]:=` **Abs[Actual-Riemann]**	The Riemann sum error.

b) Continue by typing the command lines in the left-hand column below into Mathematica in the order in which they are listed. These commands approximate our integral by the Trapezoidal Rule.

The Trapezoidal Rule requires us to weight summands to the Trapezoidal Rule approximation to our integral. We do this by creating a table (or list) in which we store the weights, and then by using the correct weight at the correct time. The weight-table is created in line 15 of the following code. It is created using a Mathematica **If** statement: the command

If[*condition, statement 1, statement 2*]

instructs Mathematica to perform *statement 1* if *condition* is true, and *statement 2* otherwise. Thus the **If** statement in line 15 says that if $n > 1$ and $n < NInts + 1$ then the associated weight is 2; otherwise it's 1.

```
In[14]:= (* Task 1b *)
In[15]:= Wt = Table[                                  Define the weight function table Wt.
           If[(n>1)&&(n<NInts+1), 2, 1],
             {n,1,NInts+1}]
In[16]:= Trapezoid = 0.0                              Initialize Trapezoid to 0.
In[17]:= For[n=1, n<NInts+1, n++,                     Compute the Trapezoidal sum.
           {Trapezoid = Trapezoid+Wt[[n+1]]*f[a+n*dx]*dx/2}]
In[16]:= Trapezoid                                    Report the Trapezoidal sum.
In[19]:= Plot[f[x], {x,a,b}, AxesOrigin->{0,0}]       Draw the graph of f.
In[20]:= Abs[Actual-Trapezoid]                        The Trapezoidal sum error.
```

c) Continue by typing the command lines in the left-hand column below into Mathematica in the order in which they are listed. These commands approximate our integral by Simpson's Rule.

Again, Simpson's Rule requires us to weight summands. The **If** statement in line 22 says that if n is even then the associated weight is 4. Otherwise if $n > 1$ and $n < NInts + 1$ then the associated weight is 2, and if $n = 1$ or $n = NInts + 1$ then it's 1.

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In[21]:= (* Task 1c *)
In[22]:= Wt = Table[                                  Define the weighting function wt.
         If[EvenQ[n], 4, If[(n>1)&&(n<NInts+1), 2, 1]], {n, 1, NInts+1}]
In[23]:= Simpson = 0.0                                Initialize Simpson to 0.
In[24]:= For[n=1, n<NInts+1, n++,                     Apply Simpson's Rule.
         {Simpson = Simpson+Wt[[n+1]]*f[a+n*dx]*dx/3}]
In[25]:= Simpson
In[26]:= Plot[f[x], {x,a,b}, AxesOrigin->{0,0}]       Draw the graph of f.
In[27]:= Abs[Actual-Simpson]                          Simpson's Rule error.
```

At this time, make a hard-copy of your typed input and Mathematica's responses. Then:

2. On each of the graphs you created in Tasks 1(a), (b), and (c), sketch and shade in the region whose area approximates $\int_{x=1}^{2} \frac{dx}{x}$ by Riemann sums, the Trapezoidal Rule, and Simpson's Rule, respectively, when $n = 6$.

3. When $n = 6$, which method — Riemann sums, the Trapezoidal Rule, or Simpson's Rule — provides the best estimate of $\int_{x=1}^{2} \frac{dx}{x} = \ln 2$? Which provides the worst?

Your lab report will be a hard-copy of your typed input and Mathematica's responses (both text and hand-drawn graphics), and your written responses.

Comments

In practice, many definite integrals cannot be evaluated in closed form, and techniques such as the Trapezoidal Rule and Simpson's Rule must be applied to find numerical approximations. Mathematica actually has built in commands that make finding such approximations quite easy. We did not use them in this project, however, since our goal was to illustrate how such methods work, not just how to use certain commands.