Exponential Growth and Decay

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Objective

To investigate the solutions of the exponential growth and decay equation.

Narrative

The number P of individuals in a population whose growth is uninhibited is described in the study of population dynamics by the differential equation $dP/dt = kP, P(0) = P_0$, where $k > 0$ is a constant. The amount A of a radioactive substance remaining after time t is described in Physics by the differential equation $dA/dt = kA, A(0) = A_0$, where $k < 0$ is a constant. These (as well as other) applications motivate the study of the exponential growth and decay equation

$$\frac{dy}{dt} = ky, \quad y(0) = y_0,$$

where k is a constant.

In this project we investigate the exponential growth and decay equation. We use the command:

DSolve[*differential equation in x and y,* **y, x**] solves the differential equation in x and y for $y = y(x)$.

As illustrated below, in using **DSolve** we be careful in specifying y by "**y[x]**".

Tasks

1. Type the command lines below into Mathematica in the order in which they are listed. These commands produce the solution to the exponential growth equation $y' = 2y$, and five solutions corresponding to various initial conditions.

`In[1]:=` **(* Your name, today´s date *)**	
`In[2]:=` **(* The Exponential Growth and Decay Equation *)**	
`In[3]:=` **(* Task 1 *)**	
`In[4]:=` **DSolve[y´[x]==2y[x], y, x]**	
`In[5]:=` **y1[x_]=0.1Exp[2x]**	In this case $y(0) = 0.1$.
`In[6]:=` **y2[x_]=0.2Exp[2x]**	In this case $y(0) = 0.2$.
`In[7]:=` **y3[x_]=0.3Exp[2x]**	In this case $y(0) = 0.3$.
`In[8]:=` **y4[x_]=0.4Exp[2x]**	In this case $y(0) = 0.4$.
`In[9]:=` **y5[x_]=0.5Exp[2x]**	In this case $y(0) = 0.5$.
`In[10]:=` **Plot[{y1[x], y2[x], y3[x], y4[x], y5[x]}, {x,-1,1}]**	

2. Repeat Task 1 for the exponential decay equation $y' = -y$.

At this point, make a hard-copy of your typed input and Mathematica's responses (both text and graphics). Then:

3. Label by hand each curve in the last graphics you created in Tasks 1 and 2: label the curve corresponding to the initial condition $y(0) = 0.1$ by "$y(0) = 0.1$", for example.

4. On the last graphics you created in Tasks 1 and 2, plot by hand the solutions to the differential equation $y' = ky$ (for the appropriate value of k) that pass through $P(1, 2)$.

5. Use the curves you drew in Task 4 to estimate $y(0.5)$.

Your lab report will be a hard-copy of your typed input and Mathematica's responses (both text and hand-labeled graphics).

Comments

A solution to the differential equation $\frac{dy}{dt} = kt$ is of the form $y = \frac{k}{2}t^2 + C$ for some constant C since

$$\frac{dy}{dt} = kt \Rightarrow dy = kt\ dt \Rightarrow y = \frac{k}{2}t^2 + C.$$

A solution to the differential equation $\frac{dy}{dt} = ky$ is of the form $y = Ce^{kt}$ for some constant C since

$$\frac{dy}{dt} = ky \Rightarrow \frac{dy}{y} = k\ dt \Rightarrow \ln y = kt + c \Rightarrow y = e^{kt+c} = e^c e^{kt} = Ce^{kt}.$$

The solutions to these differential equations are quite different; but that's appropriate since the original equations are quite different: in the first, the right-hand side is kt (k times the *independent* variable t) and in the second it's ky (k times the *dependent* variable y). The moral: we must be *very* careful in properly identifying the variables in a differential equation!