The Logistic Equation

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Objective

To investigate the logistic equation.

Narrative

The logistic equation is a differential equation used in the study of population dynamics to model the growth of a population that grows exponentially when it is small, and more slowly as it reaches the carrying capacity of its environment. (The carrying capacity of an environment is a limit to the population the environment can support, a limit that exists because of the limited resources of the environment.) If $P = P(t)$ denotes the size of a population at time t, and L is the carrying capacity of the environment supporting this population, then the logistic equation is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$

where k is a positive constant (a growth constant). The solution of the logistic equation is

$$P(t) = \frac{L}{1 + Ae^{-kt}} \quad \text{where} \quad A = \frac{L - P_0}{P_0}.$$

In this project we investigate the logistic equation and its solution.

Tasks

1. Type the command lines below into Mathematica in the order in which they are listed. These commands check the soultion to the differential equation described in the Narrative.

```
In[1]:= (* Your name, today´s date *)
In[2]:= (* The Logistic Equation *)
In[3]:= (* Task 1 *)
In[4]:= A = (L-P0)/P0
In[5]:= P[t_] := L/(1+A*Exp[-k*t])
In[6]:= P´[t]-k*P[t]*(1-P[t]/L)
In[7]:= Simplify[%]
In[8]:= P[0]
In[9]:= Simplify[%]
```

2. Continue by typing the following command lines into Mathematica in the order in which they are listed. These commands identify solutions to the logistic equation for the case in which $L = 2$ and $k = 2$ corresponding to the initial values of $P_0 = 0.1, P_0 = 0.75, P_0 = 1.5, P_0 = 2.0, P_0 = 2.5$, and plot these solutions in one graphic.

```
In[10]:= (* Task 2 *)
In[11]:= P1[t_] := P[t]/.{L->2, k->2, P0->0.1}
In[12]:= P2[t_] := P[t]/.{L->2, k->2, P0->0.75}
In[13]:= P3[t_] := P[t]/.{L->2, k->2, P0->1.5}
In[14]:= P4[t_] := P[t]/.{L->2, k->2, P0->2.0}
In[15]:= P5[t_] := P[t]/.{L->2, k->2, P0->2.5}
In[16]:= Plot[{P1[t],P2[t],P3[t],P4[t],P5[t]}, {t,0,4}, PlotRange->{0,3}]
```

At this point, make a hard-copy of your typed input and Mathematica's responses (both text and graphics). Then:

3. Label by hand the graph of each function in the graphic you created in Task 2: label the graph corresponding to the initial condition $P(0) = 0.1$ by "$P(0) = 0.1$", for example.

4. On the graphic you created in Task 2, plot by hand the solution to the differential equation $P' = P(L-P)$ that passes through the point $P(2, 1)$.

5. If $P(t) < L$:

a) what does the logistic equation imply about the sign of dP/dt? is P increasing or decreasing? Explain (algebraically) how you arrive at your answer.

b) explain (in the context of population growth) why P should increase to L as t gets large.

6. If $P(t) > L$:

a) what does the logistic equation imply about the sign of dP/dt? is P increasing or decreasing? Explain (algebraically) how you arrive at your answer.

b) explain (in the context of population growth) why P should decrease to L as t gets large.

Your lab report will be a hard-copy of your typed input and Mathematica's responses (both text and hand-labeled graphics), together with your hand-written responses.