Euler's Method

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Objective

To investigate Euler's method.

Narrative

Not all differential equations can be solved analytically. There are techniques, however, such as Euler's Method, that allow you to solve such equations numerically.

Suppose, for example, we want to find the value of a solution $y = f(x)$ to the equation

$$\frac{dy}{dx} = F(x, y)$$

at $x = b$ given its value $f(a)$ at $x = a$. (For simplicity, we assume that $a < b$.) Since, for small values of h,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \approx \frac{f(x+h) - f(x)}{h}$$

we divide the interval $[a, b]$ into N subintervals of equal length $h = (b - a)/N$, for some sufficiently large integer N, and let $x_n = a + i * h, i = 0, \ldots, N$. Then, for each n,

$$f'(x_n) \approx \frac{f(x_n + h) - f(x_n)}{h} = F(x_n, y_n)$$

where $y_n = f(x_n)$, so

$$f(x_{n+1}) = f(x_n) + hF(x_n, y_n) \quad \text{or} \quad y_{n+1} = y_n + hF(x_n, y_n).$$

This equation provides us with a mechanism from marching from (the known value) $f(a) = f(x_0)$ to $f(b) = f(x_N)$: Given x_0 and $f(x_0)$ it allows us to compute $f(x_1)$; given x_1 and now knowing $f(x_1)$ (at least approximately), we can compute $f(x_2)$; given x_2 and now knowing $f(x_2)$ (at least approximately), we can compute $f(x_3)$; etc.:

$$\begin{aligned} y_0 &= f(x_0) = f(a) \\ y_1 &= f(x_1) = f(x_0) + hF(x_0, y_0) \\ y_2 &= f(x_2) = f(x_1) + hF(x_1, y_1) \\ \vdots &= \vdots \end{aligned}$$

Eventually we will be able to compute $y_N = f(x_N) = f(b)$.

Task

1. Type the command lines below into Mathematica in the order in which they are listed. They solve the differential equation $y' = x + y$ with the initial condition $y(0) = 0$, analytically and evaluate this solution when $x = 1$.

```
In[1]:= (* Your name, today´s date *)
In[2]:= (* Differential vs Difference Equations *)
In[3]:= (* Task 1 *)
In[4]:= Solution = DSolve[{y´[x]==x+y[x],y[0]==0},y,x]
In[5]:= f[t_] := (y/.Solution[[1,1]])[t]
In[6]:= f[t]
In[7]:= N[f[1],6]
```

2. Continue by typing the following command lines into Mathematica. They apply Euler's Method to $y' = x + y$ assuming that $a = 0$, $f(a) = f(0) =$, $b = 1$, and $N = 10$.

```
In[8]:= (* Task 2 *)
In[9]:= {a=0.0, b=1.0, NInts=10}
In[10]:= {x[0]=a, y[0]=0, h=(b-a)/NInts}
In[11]:= For[n=0, n<NInts+1, n++,
           {x[n+1]=x[n]+h, y[n+1]=y[n]+h*(x[n]+y[n])}]
In[12]:= {n,y[n]}
```

Note the discrepancy between the numerically computed values of f and the analytically computed values.

3. Repeat Task 2 with a values for N of 100, 1000, 10,000, and 100,000. Make a table containing these values (and the value for $N = 10$), and attach this table to a hard-copy of your typed input and Mathematica's responses: it will serve as your lab report.

Comments

1. In this project, we solved a differential equation analytically, graphically, and numerically, and were able to compare our solutions. In many cases, it is difficult — or even impossible — to solve a differential equation analytically, and in this case the only alternatives are graphical and numerical solutions.

2. While Euler's Method is good for illustrating the general approach to numerically solving differential equations, it does not generally yield extremely accurate results. Other numerical techniques exist, and these are studied in courses on Numerical Analysis.