The Second Derivative Test Again

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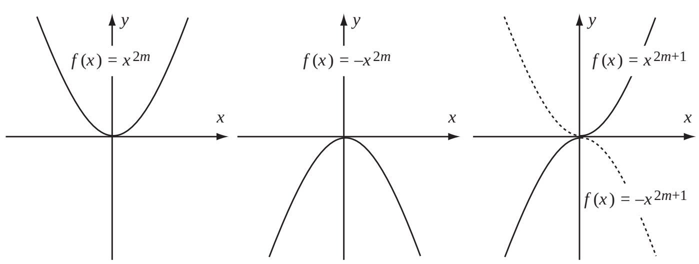
Objective

One of the major limitations of the Second Derivative Test for functions $f : R \to R$ of one variable is that if $f'(0) = 0$ (so $x = 0$ is a critical number for f) and $f''(0) = 0$ then the Second Derivative Test provides no information about whether $f(0)$ is a relative maximum, a relative minimum, or neither. In this project we illustrate how we can get more information using concepts related to Mclaurin series.

Narrative

Our analysis is based on the Mclaurin series and two facts. The first fact is that if M and N are integers for which $M > N$ then $|x|^M < |x|^N$ whenever x is a real number close to 0, so the behavior of x^N dominates that of x^M; that is, x^M is small in comparison to x^N. The second is that if n an integer and:

1. $f(x) = x^{2n}$ then f is positive for all $x \neq 0$, so $f(0) = 0$ is a relative minimum,
2. $f(x) = -x^{2n}$ then f is negative for all $x \neq 0$, so $f(0) = 0$ is a relative maximum, and
3. $f(x) = \pm x^{2n+1}$ then the graph of f is positive for some values of x near $x = 0$ and negative for others, so $f(0)$ is not a relative extremum.



The Mclaurin series expansion for a real-valued function $f : R \to R$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \ldots.$$

Thus if $f'(0) = 0$ and $f''(0) = 0$, then

$$f(x) = f(0) + \frac{f'''(0)}{3!} x^3 + \ldots.$$

Since x^M is considerably smaller in magnitude than x^3 for all $M > 3$ and all x sufficiently close to 0, it follows that

$$f(x) \approx g(x) = f(0) + \frac{f'''(0)}{3!} x^3$$

for x sufficiently close to 0. Thus, since $f(0)$ only affects the vertical shifting of the graph of f, not the shape of the graph of f,

if $f'(0) = 0, f''(0) = 0$, and $f'''(0) \neq 0$ then $f(0)$ is not a relative extremum.

If $f'(0) = 0, f''(0) = 0$, and $f'''(0) \neq 0$ then

$$f(x) = f(0) + \frac{f^{(iv)}(0)}{4!} x^4 + \ldots$$

and it follows that

$$\textit{if } f'(0) = 0, f''(0) = 0, f'''(0) = 0, \textit{ and } f^{(iv)}(0) > 0 \textit{ then } f(0) \textit{ is a relative minimum, and}$$
$$\textit{if } f'(0) = 0, f''(0) = 0, f'''(0) = 0, \textit{ and } f^{(iv)}(0) < 0 \textit{ then } f(0) \textit{ is a relative maximum.}$$

This type of argument can be continued.

Example: If $f(x) = x^2(e^{x^2} - 1)$ then $f'(0) = 0$ and $f''(0) = 0$, so the Second Derivative Test provides no information. However, using Mathematica we find that $f'''(0) = 0$ and $f^{(iv)}(0) = 24 > 0$; thus $f(0)$ is a relative minimum.

Tasks

For each of the following functions, $f'(0) = 0$ and $f''(0) = 0$, so the Second Derivative Test provides no information. Use Mathematica to compute enough derivatives of f at $x = 0$ to determine whether $f(0)$ is a relative maximum, a relative minimum, or neither. Make sure your reasoning is clear, and that your answers are stated clearly.

1. $f(x) = 2x^6 - 6x^4$
2. $f(x) = 3x^4 - 4x^3 + 6$
3. $f(x) = 3x^5 - 5x^3$
4. $f(x) = 8x^2 - 2x^4$
5. $f(x) = 1 - \cos x^2$
6. $f(x) = x \ \mathrm{Tan}^{-1} x^2$
7. $f(x) = x - \ln(1 + x^2)$
8. $f(x) = e^{x^3} - 1$

Comments

The technique presented above can easily be extended to values c of x for which $f'(c) = 0$ and $f''(c) = 0$, other than $c = 0$. The only changes that must be made are: 1) shifting attention from the Mclaurin series expansion for f to the Taylor series expansion for f about $x = c$, and 2) shifting attention from values of x that are close to 0 to values of x that are close to c.